LIBERTY UNIVERSITY

When Life Gives You Circles: Calculating the Square Root of Two

Isaiah Mellace, Joshua Kroeker

Abstract

In this poster, we derive a method of approximating the square root of two. We do this by constructing a geometric figure that has inscribed circles lying across the diagonal of a square. By adding the radii of these circles, we can find the length of the diagonal, which will be the square root of two if the square had side length one. This project uses trigonometry, recursion, the binomial theorem, and a lot of algebra to arrive at the following equation:

$$\sqrt{2} = \lim_{x\to\infty} \frac{-3 + 6\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}8^j3^{i-2j}}{3 + 4\sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}8^j3^{i-2j}}$$

We find that the formula gives an approximation that is about 3 decimal places more precise for every iteration of this sequence. While other methods exist for calculating square roots more efficiently, geometric methods such as these are scarcely found and can apply to other problems. Specifically, these methods have potential applications in finding square roots generally, approximating pi, and describing other irrational numbers. Although space filling techniques have been used to find values, this method may be better as it uses circles. Additionally, it should be noted that this function has some properties which may lead to fast calculations using a program with optimizations. Namely, the 8s can be solved using bit manipulation and constructing Pascal's triangle through recursion can eliminate the use of factorials.

Introduction

Efficient algorithms for computing square roots are essential for various applications, including numerical simulations, graphics rendering, cryptography, and signal processing. As such, there are many methods that have be discovered to approximate these numbers quickly and accurately. We do just this by claiming that the diagonal is approximated by diameters of the circles lying on it as seen in **Figure 2**. By solving for the radii of each circle and adding them up, we approach the length of the diagonal. The following is a demonstration of how this method arrived at the equation listed above.

Methods

To now find the radii, we noticed that the radius of the original circle is equal to the sum of the vertical magnitudes of the circles' diameter along the diagonal and the final unknown circle's radius, as seen in **Figure 3**. Using some algebra and induction we find the value of the radii of the preceding circles. Combining this with the equation in **Figure 5**, we get **Figure 6**. From here we have the issue of defining the root with itself, but this can be resolved by grouping all terms that have square roots on one side as seen in **Figure 7**. A general equation then arises through categorizing the terms with Pascal's Triangle as is seen in **Figure 8**. Putting this all together we arrive at **Figure 9**.

$$\sqrt{2} = \lim_{x\to\infty} \frac{-3 + 6\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}8^j3^{i-2j}}{3 + 4\sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}8^j3^{i-2j}}$$

Figure 1. The resultant equation for the square root of two

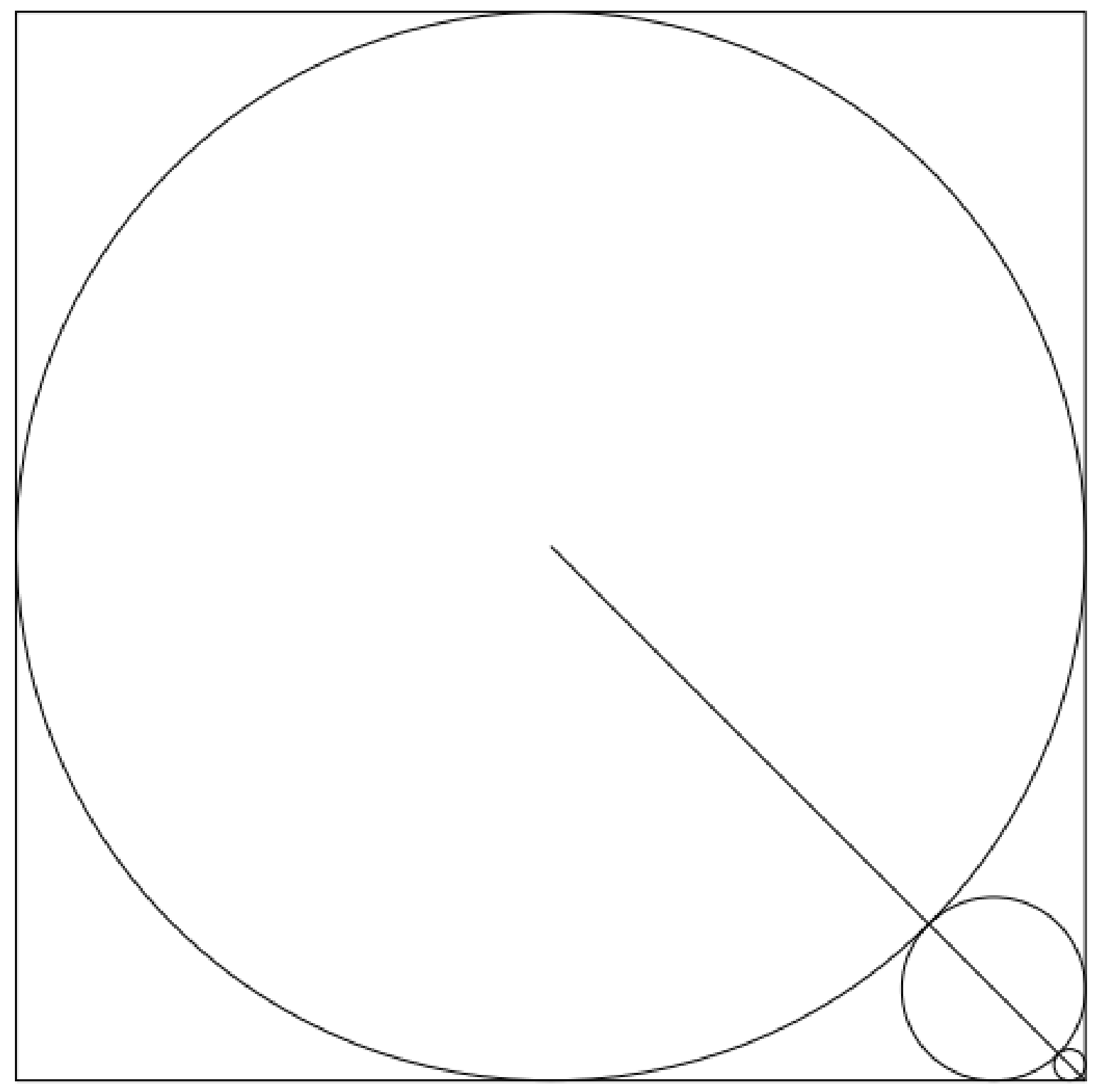
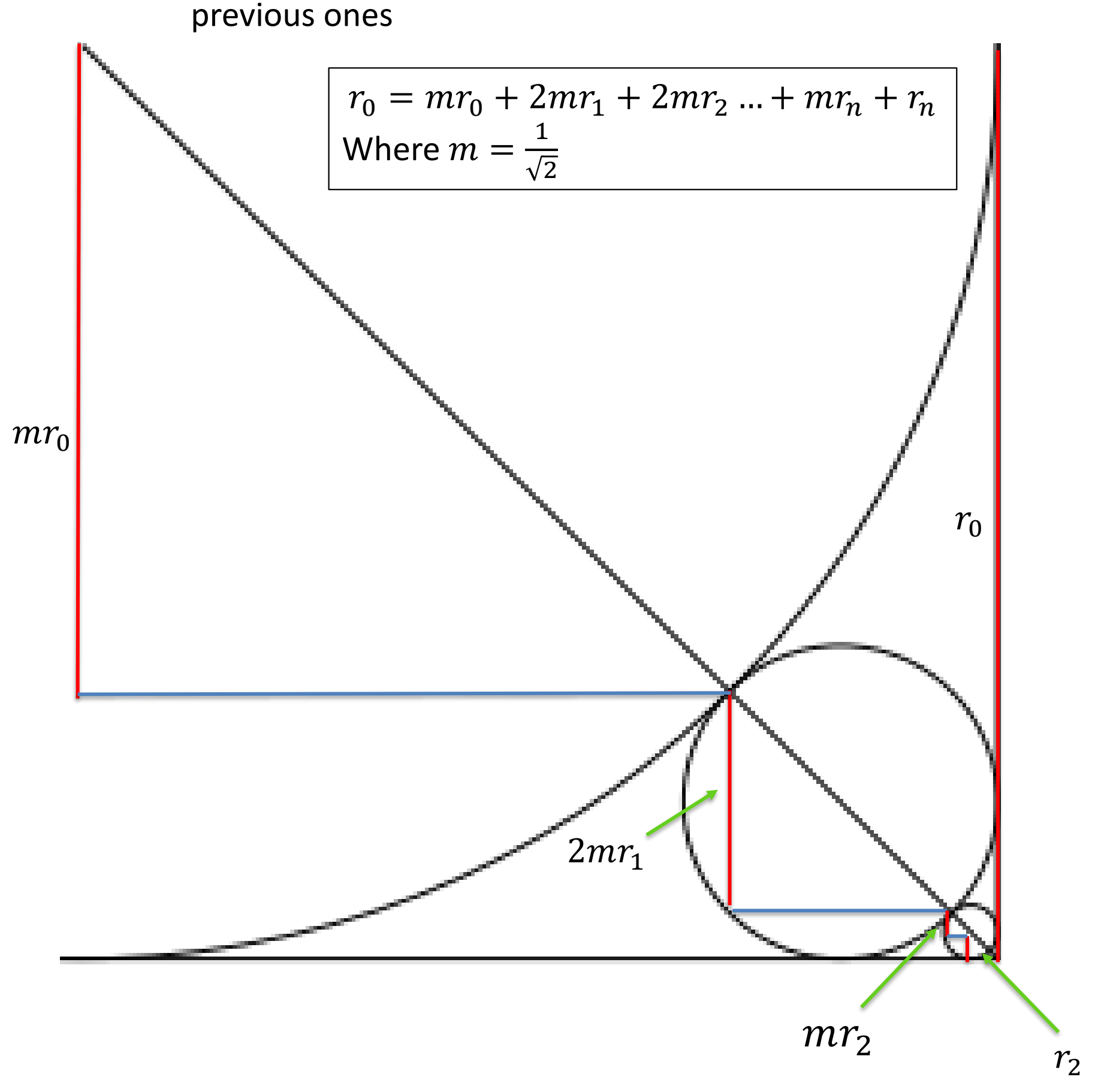


Figure 2. The first three circles lying on the diagonal of a square

Figure 3. Solving for the next circle using the previous ones

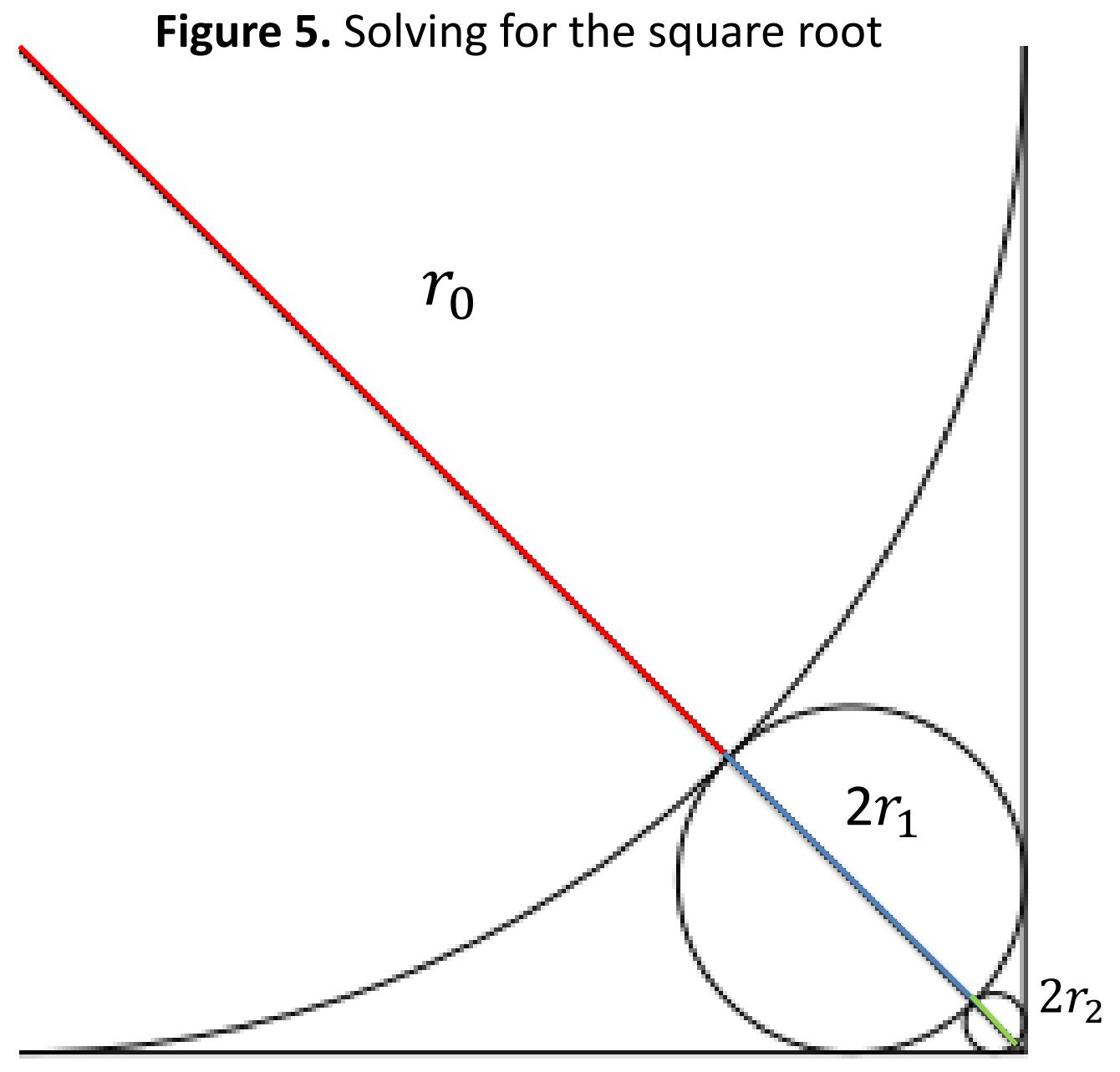


$r_0 = mr_0 + 2mr_1 + 2mr_2 \ldots + mr_n + r_n$
Where $m = \frac{1}{\sqrt{2}}$

Finding r_1	$r_0 = mr_0 + mr_1 + r_1$ $r_1 = r_0\frac{1-m}{1+m}$
Finding r_2	$r_0 = mr_0 + 2mr_1 + mr_2 + r_2$ $r_2 = r_0\frac{1-m-2mr_1}{1+m}$ $r_2 = r_0\left(\frac{1-m}{1+m}\right)^2$
Generally,	$r_n = r_{n-1}\left(\frac{1-m}{1+m}\right)$ or $r_n = r_0\left(\frac{1-m}{1+m}\right)^n$

Figure 4. Solving for the radius

Figure 5. Solving for the square root



$\sqrt{2} \approx r_0 + 2r_1 + 2r_2 \ldots$
Assuming the square has a side length of 2

$$\sqrt{2} \approx r_0 + 2r_0\left(\frac{1-m}{1+m}\right) + 2r_0\left(\frac{1-m}{1+m}\right)^2 + \cdots + 2r_0\left(\frac{1-m}{1+m}\right)^n$$

$$\sqrt{2} \approx \lim_{x\to\infty} r_0\left(1 + 2\sum_{n=1}^{x}\left(\frac{1-m}{1+m}\right)^n\right)$$

Now because the square has side length 2, $r_0 = 1$.

Also $\frac{1-m}{1+m} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = 3 - 2\sqrt{2}$

$$\sqrt{2} = \lim_{x\to\infty} 1 + 2\sum_{n=1}^{x}\left(3-2\sqrt{2}\right)^n$$

Figure 6. Combining equations from **5.** and **4.**

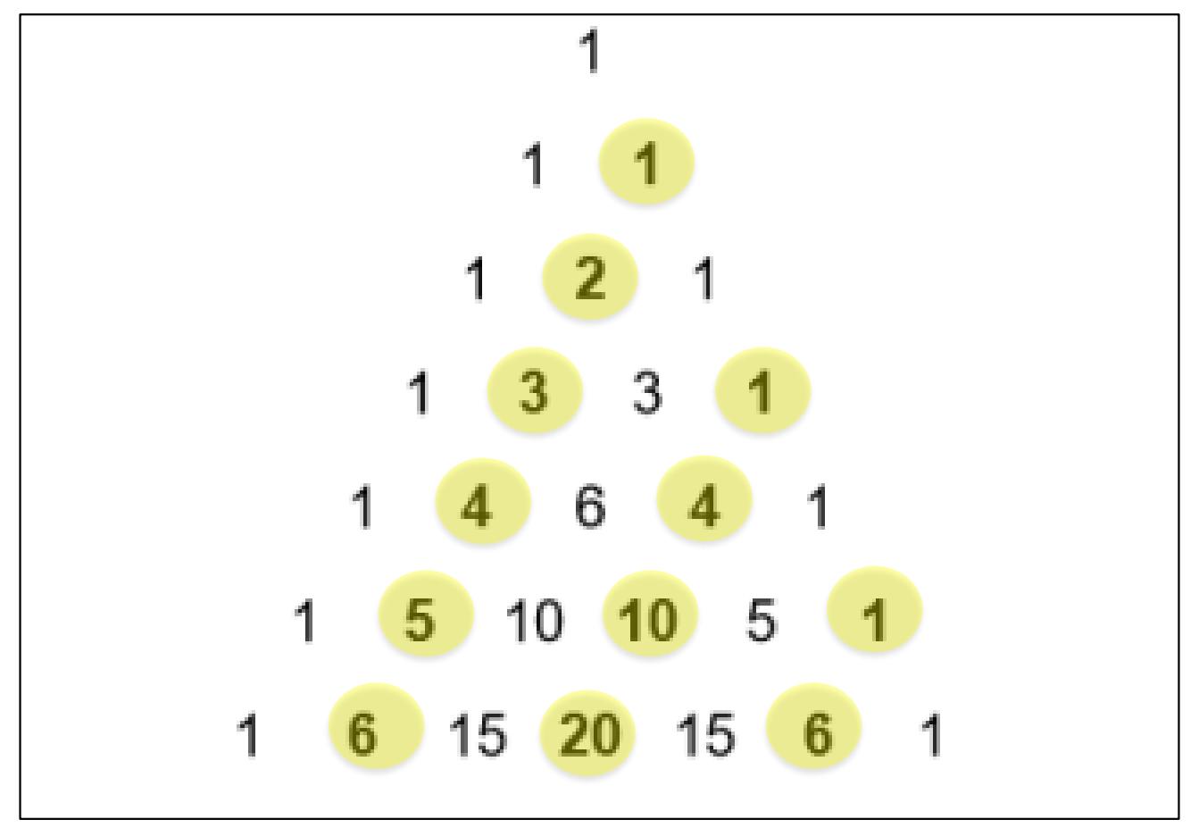
$$\sqrt{2} \approx 1 + 2\sum_{n=1}^{2}\left(3-2\sqrt{2}\right)^n$$

$$\sqrt{2} \approx 1 + 2\left(3 - 2\sqrt{2} + 9 - 12\sqrt{2} + 8\right)$$

$$\sqrt{2}(1 + 4 + 24) \approx 1 + 6 + 18 + 16)$$

$$\sqrt{2} \approx 1.413793$$

Figure 7. An example of grouping the roots



The terms we are interested in moving are those that have $-2\sqrt{2}$ as odd power. By grouping the highlighted terms above we get

$$\sum_{n=0}^{x}\left(3-2\sqrt{2}\right)^n =$$
$$\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}(-2\sqrt{2})^{2j}3^{i-2j} + \sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}(-2\sqrt{2})^{2j+1}3^{i-(2j+1)}$$

Figure 8. Grouping terms using the binomial theorem

Note: -1 is present because the sequence starts from $n = 0$ instead of $n = 1$

$$\sqrt{2} = \lim_{x\to\infty} 1 + 2\left(-1 + \sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}(-2\sqrt{2})^{2j}3^{i-2j} + \sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}(-2\sqrt{2})^{2j+1}3^{i-(2j+1)}\right)$$

$$\sqrt{2}\left(1 - \frac{4}{3}\lim_{x\to\infty}\sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}8^j3^{i-2j}\right) = -1 + 2\lim_{x\to\infty}\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}8^j3^{i-2j}$$

$$\sqrt{2} = \lim_{x\to\infty} \frac{-3 + 6\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}8^j3^{i-2j}}{3 + 4\sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}8^j3^{i-2j}}$$

Figure 9. The Final Form

Results

Results:
While the final function is written mathematically, it would be far more efficient to go back to **Figure 8** and implement the idea into a computer program. Nonetheless, upon writing the function into Mathematica, some fascinating features were discovered with the algorithm:

Let the function be defined as $f(x) = \frac{-3+6\sum_{j=0}^{x}\sum_{i=2j}^{2x+1}\binom{i}{2j}8^j3^{i-2j}}{3+4\sum_{j=0}^{x}\sum_{i=2j+1}^{2x+1}\binom{i}{2j+1}8^j3^{i-2j}}$

1. Error
To find how much more precise a function is from term to term you simply find the error from one term then divide it by the error from the next

$$E(x) = \frac{\sqrt{2} - f(x)}{\sqrt{2} - f(x+1)}$$

This function approaches 1154. Upon evaluating $\log_{10} 1154$ we discover that the function is about 3.062205809 more digits precise every iteration

2. Actual Values
While the previous number is just found analytically, it is necessary to check through to see if results look that way. For the value of $x = 1000$, $f(x)$ produces

89132976959472716469343673065372374311325322549818418084257824046585654409934271340700076248627213969599398108620485132
34978409820544654135770697208337649794236800797724346897792926219671955240911857614702750872407809572804962813597318164
22850077984317970606002461738701320455114868811301054903719874302786970112275055065387934136365357589919853458904623646
34800729282372464386752229779142588988883454384896323203834780938379275759784509574665997200140215077749461010287089309
63287304064037240878455856000802916424822989157650038923188262788310271633773426744219455762581564498934107568910794469
47509052376536999019672636806915739309837952547906176985188749224825791574110267754300231857483380080857040033136722173
35370253875469780432241761301906074309433734605886074205888897902801601523232527390992286334372689916602758053397806282
24541046829302537578867924267837831290130395268484836244203083792598844188543774818332111430961143113925183819698833325
61303806475260480040206151636733439983556550605005028185315615492419446375332876211311659540079589024433956790802120016
36820425434055258835873521565842237397310804430535438915860830037960426881468413910685484879589741003551648200009089349
48815508640316544576720506537520233970741862612718613161417751864992208598870115539131265665993510212334842660590331919
65076366932666198133560263504930524129877358384357319893290259956282168908121161711552267400761338965956673154237663822
75266057097310367164863093450396601424965118276846877208407637963152073508055341089684324463579853614279

as a numerator and

63026532435387456194230424738707994173378135766140251243029105614110507264708444239269253450906299605916903330077993742
16548455424190097984724681879407180267399851818441179742181687876576848650307665931308467107909610847768382681818638085
96209709842708173551439309896038738648387846323648977999417012972181858183260494452387633110403387496975965761845280078
20373798877815272309932288995394965064416543684660002056147627546505035847719211018931184183354848713846158131128299281
77870881927977964678647643015252024201691963596297782857080130524735298405839327136577731815053690463698837502455783893
80132031899088790891997533684514304629554313923980098029228998036135258205567446955531626286529646144643492217695387251
294947593040471103876991758272595713747193931911379061089071118553983680816379129570830858116724734628578706554744719973
51674434312352748852393107929327036060632133540004854037191989315489355174727437539088282255937426739675286273091814487
75607569752945419538575347481377860953419455810198912435274022307910581869070680276410490946539129692500159238202186985
18120434268474204937652138288122626096968363903407240909099659386232482754894956757716371431528970145593903423974029952
16574559267158428864722997574070069545154106662853505541516020085023704846384929477052340058835896271295898237338727448
61769699402027352305076955026867914253920715557899154509746765543917375588653469159556181816896605209350634327814798373
9587069705461232594206523566650956011760293128518461030965551633535566149250635499497511366945775191589

as a denominator; Which produces an error of $8.900372405\ldots \cdot 10^{-3065}$. Not bad!

Note: The initial values of this sequence gave approximations that were about 4 more decimal places precise each iteration for some fascinating reason.

Future Work

1. Generalizing this method to find any square root function.
2. Experiment with various configurations of shapes to approximate transcendental numbers*.
3. Find a more rapidly converging sequence by using a "tighter" fit such as a rotated ellipse in this square.
4. Calculating the square root of two to millions of digits
5. Expand the method to multiple dimensions (Spheres and Cubes)

*There is a fair chance this cannot be done as transcendental numbers cannot be constructed with algebraic numbers alone