Gödel's Incompleteness Theorems

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Abstract

The Incompleteness Theorems of Kurt Gödel are very famous both within and outside of mathematics. They focus on independence and consistency within mathematics and hence a more thorough understanding of these is beneficial to their study. The proofs of the theorems involve many ideas which may be unfamiliar to many, including those of formal systems, Gödel numbering, and recursive functions and relations. The arguments themselves mirror the Liar's Paradox in that Gödel constructs a statement asserting its own unprovability and then shows that such a statement and its negation must both be independent of the system, otherwise the system is inconsistent. We then proceed to survey various interpretations of the Incompleteness Theorems, focusing on potential misapplications of the theorems.

Introduction

The field of mathematics can seem to one with beginning or intermediate levels of understanding to be a source of objective truth. It has often been described as the language of the universe itself. Regardless of one's beliefs on such grandiose statements, the truth is that mathematics in its primarily axiomatic and deductive modern form is fundamentally limited in some crucial ways. Two of these limitations are usually referred to as the ideas of incompleteness and consistency. The understanding of these two limitations is largely due to the pioneering work of Kurt Gödel, an Austrian mathematician who published his revolutionary results in 1931. At that point, the ramifications of these results were not fully understood. Since the publishing of Gödel's original paper, many other mathematicians have clarified details omitted by Gödel and built upon his results. However, in discussing incompleteness and consistency, we will focus first on understanding some of the mathematical landscape leading up to 1931. Afterwards, we will discuss the main techniques used by Gödel in his paper, outline his major results and proofs, and analyze the ultimate implications of Gödel's results, collectively referred to as Gödel's Incompleteness Theorems.

The Conflicts with Consistency and Undecidability

In the past century, the axiomatic method of doing mathematics has grown exponentially in popularity. The cornerstone of this approach for any mathematical field is that certain basic statements, known as axioms, are assumed to be true. Further results are then deduced directly from the given axioms. Often these axioms are statements that are widely accepted to be true so that no one may reasonably object to their assumption, though this is not always the case.

For a little over a millennium, the only major mathematical field based on the axiomatic approach was geometry, which itself was largely based on Euclid's *Elements*. In this work, Euclid assumed five basic axioms as the foundation for his geometric system. The final axiom Euclid included, however, was controversial among mathematicians, for it seemed a very unjustified assumption [1]. This controversial axiom is equivalent to the claim that for any line ℓ and any point P not on ℓ, there is only one line m through P parallel to ℓ. Upon observing the physical world, this seems as though it would have to be true in geometry. Yet, after nearly two thousand years of failed attempts to prove this parallel postulate, it was ultimately shown that Euclid's final axiom is in fact independent of his other four axioms [1]. Another way to describe this situation is that Euclid's final axiom is undecidable given his other four axioms; it is neither provable nor disprovable.

Euclid's parallel postulate is just one of many different examples of the occurrence of independent statements in formal mathematics. Another is the Continuum Hypothesis within set theory. The Continuum Hypothesis is equivalent to the claim that there is no set with cardinality strictly between that of $\mathbb{N}$ and $\mathbb{R}$. Thanks to the work of Kurt Gödel and Paul Cohen, we now know that the Continuum Hypothesis is independent of the modern axioms of set theory [2]. Euclid's postulate and the Continuum Hypothesis represent a class of statements from across the universe of mathematics that are all independent of their respective axioms.

The existence of such independent statements invites many different questions. First, does the fact that these statements are independent of their respective axioms bear any weight on their possible truth or falsity? This question necessitates a discussion of what "truth" really means in mathematics, which will be briefly addressed later. Additionally, the two examples of independent statements provided thus far are of the quality that it is quite difficult to imagine a circumstance in which these statements are not true. Is our intuition at fault for this? Certainly with more advanced topics in mathematics one's natural intuition can prove counterproductive to the reality of the situation and validity of some results. Or does this portend once again the idea that ultimate truth may not be the goal of formal mathematics?

A separate but closely related issue that has concerned mathematicians for centuries is that of consistency. A deductive system is considered consistent if, given its starting axioms and rules of inference, for a given statement p it is impossible to prove both p and its negation $\neg p$ within that system. More directly, a consistent system cannot give rise to a contradiction (assuming axioms and rules of inference are correctly applied). Mathematicians through history have gone through great pains to convince themselves of the validity of their results. However it is a somewhat disturbing notion that somewhere in the system one is working in there may be a hidden contradiction lurking which, upon discovery, leads to the collapse of many major results. Thus it would be beneficial to prove in absolute terms that a deductive system is consistent and hence that no contradiction will ever be found in such a system. This was the hope of the great mathematician David Hilbert, who was also a strong proponent for the axiomatic method in general [3].

As concerns about consistency became more prominent, some proofs were discovered to establish the consistency of a certain deductive system, though never quite possessing the absolute nature desired. In some cases, it is possible to translate the question of whether a given deductive system is consistent to an arithmetical system and thus prove the deductive system's consistency using methods from the arithmetical system [3]. However, these proofs are reliant upon the assumption that the arithmetical system itself is consistent, which has not been proven. While this seemed to be a step in the right direction, it still required the assumption of consistency at some point. The hope of Hilbert and other mathematicians was that one might be able to prove a system's consistency within the language of the system itself and without any other assumptions regarding consistency [3]. Such a proof would be considered an *absolute* proof of consistency, as opposed to the relative consistency proofs that had been demonstrated. Another characteristic of such an absolute proof, as defined by Hilbert, is that it would be *finitistic* in its reasoning [3]. This means essentially that there must be no steps in the proof that refer to an infinite sequence of steps or utilize an argument based on similarly infinite ideas. As Gödel was to show in his paper, Hilbert's program and ultimate goals with consistency were impossible.

Before moving on, let us establish briefly the immediate results proven by Gödel. First, in any sufficient system of arithmetic that is assumed consistent, there exist true but unprovable propositions of arithmetic; in essence, arithmetic is incomplete [4]. Often called the First Incompleteness Theorem, this result was revolutionary to many once understood. Note that by a sufficient system of arithmetic, we mean a system with the means necessary to express the positive integers and zero, as well as the operations of addition and multiplication. This is important to note because every proposition in an arithmetical system utilizing only addition is in fact decidable [5]. Second, if a formal system is assumed consistent, then it cannot prove its own consistency [4]. This is usually referred to as the Second Incompleteness Theorem and seems to have been a surprising consequence of Gödel's main goal of proving undecidability in arithmetic.

Formal Systems in Metamathematics

Crucial to the understanding of most of Gödel's arguments in his monumental paper is the idea of a formal system. First, we should note that since Gödel's ultimate results amount to statements *about* mathematical systems, these findings and arguments fall under the field of metamathematics. Metamathematics is at its heart concerned with discussing and proving things about the structure and the syntax of mathematics rather than specific things expressed within a given mathematical system [5]. For example, in algebra the proposition that every subgroup of a cyclic group is itself cyclic is a mathematical statement. However if we were to let p represent this proposition, then the claim that p is provable would be a metamathematical statement. Such a statement asserts a property inherent to the concept of deductive reasoning in math and not one unique to any given system. A very useful tool in expressing metamathematics is what Gödel refers to as a *formal system*, sometimes called a "calculus" by other authors [4], [5]. Formal systems provide a way of understanding and interpreting deductive systems in mathematics through the use of symbolic logic.

Formal systems heavily rely upon the machinery of propositional logic discussed in most transitional upper-level mathematics courses. Many common symbols from propositional logic can mean the same thing in a formal system, for example with $\vee$ representing the logical "or" and $\neg$ representing the negation. However formal systems only use this notation as a starting point. A formal system at its core is composed of *formulas*; by this we mean a string of logical variables and operators that, when combined form a coherent statement with a definitive truth value [4], [5].

For example, if we let p and q represent two different sentences within any given deductive system, a formula in this instance might take on the form of $[(\neg p) \Rightarrow q]$. It is these types of formulas that compose any formal system. We must note that Gödel restricted his class of functions to those which are "meaningful" in the context of a formal system [4, p. 38]. For instance, $)\vee \Rightarrow \forall\neg)$ is a finite string of logical operators. However, these particular operators combined in such a way are not meaningful. Thus, Gödel implicitly excludes any formulas of this type from his considerations.

For these formal systems to give any sort of insight into the nature of the deductive system in question, one must develop several things. First, the initial formulas of the system must be constructed to logically correspond to the axioms of the deductive system. Then, the rules of inference used to apply axioms and prove theorems within the system must be translated into rules for symbolic manipulation in the formal system [5]. Thus a proof of a formula within a formal system amounts to a *finite sequence of formulas* obtained by successive application of the translated rules of inference. If this is performed correctly, it yields a one-to-one correspondence between the deductive system and the formal system. In this way, a proof of a formula amounts to a proof that the theorem that is the interpretation of the formula within the deductive system *can be proved* using the axioms and rules of inference [5]. Constructing a formal system P to suit his needs was thus one of the first things Gödel set out to do.

Mapping and Gödel Numbering

Another concept that is critical to Gödel's argument and one that he himself devised is a specific way of assigning numbers to formulas, called Gödel numbering. As established already, Gödel's results are really metamathematical, or rather meta-arithmetical, statements and so Gödel's efforts are greatly helped by the application of his formal system. However, one of Gödel's several strokes of genius is his idea of mapping formulas and sequences of formulas within his formal system directly into an arithmetical system. In this way he would be able to treat questions of *meta-arithmetic* within the language of arithmetic itself. This corresponds in some way to the work of Descartes in developing a coordinate geometry system [6]. These coordinates allowed geometric truths and proofs to be explored using algebraic methods. In the same way, Gödel's mapping allows meta-arithmetical proofs to be discussed using arithmetical methods [6].

In addition to this idea, Gödel created an ingenious method of actually mapping formulas onto numbers that relies heavily on the power of prime numbers and the fundamental theorem of arithmetic. First, he distinguished between constant signs and variables, and to each of these were assigned specific numbers. The constants of his formal system and numbers assigned to them are as follows:

$$''0'' \leftrightarrow 1, ''f'' \leftrightarrow 3, ''\neg'' \leftrightarrow 5, ''\vee'' \leftrightarrow 7, ''\forall'' \leftrightarrow 9, ''('' \leftrightarrow 11, '')'' \leftrightarrow 13$$

where in this case f is the "successor of" function. In other words $1 = f0$, since 1 is the successor of 0 in whole number arithmetic [4]. As for the variables, Gödel further distinguished them into three categories:

1. Variables of the first kind, which represent individual numerals
2. Variables of the second kind, which represent full sentences, or formulas within the formal system

3. Variables of the third kind, which represent predicate expressions, or rather expressions that indicate relationships or properties of numbers, such as “Less than.”

Thus a variable of the nth kind translates to p^n, where $p > 13$ is a prime [4].

Thus far, we have determined a way to map individual components of formulas into unique numbers and thus we have a way to express them arithmetically. However, it would be beneficial to map entire formulas into single numbers. Gödel’s method to do this is rather straightforward. Suppose we have a formula composed of n elements with respective Gödel numbers $m_1, m_2, ..., m_n$. The Gödel number corresponding to the complete formula then is the product of the first n primes raised to the power of the Gödel numbers of each respective component. In other words, the Gödel number g of the formula would be given by

$$g = p_1^{m_1} \times p_2^{m_2} \times ... \times p_n^{m_n}$$

where $p_1, p_2, ..., p_n$ are the first n primes [4]. In the same manner, we can also map a sequence of formulas forming a proof within the formal system to a single number. We will refer to such a sequence of formulas in the formal system simply as a proof of a formula.

Let us now pause and reiterate the essence of what Gödel has accomplished with his mapping and its importance. By mapping variables of each kind into unique numbers, we are then able to express simple statements about a system in the language of the system. By then mapping formulas and sequences of formulas into unique numbers, we are able to express essentially any possible statement and a proof of such statement *about* arithmetic within the *language* of arithmetic [6]. So relationships between formulas, and thus the statements they represent, will correspond to specific arithmetical relations between their respective Gödel numbers [3]. Finally, a detail crucial to this discussion is that Gödel's mapping forms a *bijection*, or a one-to-one correspondence, between the formal system and the arithmetical system. This is due to the fundamental theorem of arithmetic, stating that any positive integer greater than 1 can be uniquely expressed as the product of primes to varying nonnegative integer exponents. Thus, given a formula or sequence of formulas, a unique Gödel number representing it can be found *and* given any Gödel number, the unique formula or constant sign corresponding to it can be found [3]. However it must be noted that this one-to-one correspondence is between the set of formulas in the formal system and a subset of the positive integers; not every positive integer is a Gödel number [5].

The nature of Gödel's remarkable yet straightforward bijection leads us to another critical observation: relationships between formulas and proofs in the formal system indicate the existence of arithmetical relationships between their corresponding Gödel numbers [5]. Furthermore, this allows the relationships between formulas described in plain language to be used to describe the relationships between corresponding Gödel numbers. For example, let us suppose that a sequence of formulas provides a proof of a specific formula. Now if we let x be the Gödel number of the sequence of formulas and y the Gödel number of the proven formula, we can now be sure that a specific arithmetical relationship exists between x and y. We can thus describe this relationship as "x provides a proof of y." Gödel denotes this statement in his paper by xBy [4]. In his paper, extensive use is made of such shorthand notation to represent relationships. However, much of this notation will not be defined or used here for the sake of clarity and brevity.

Recursive Functions and Definitions

A final aspect of Gödel's argument that deserves discussion is the notion of recursiveness in mathematics. A *recursive definition* is one in which the first element of a sequence is defined as well as a rule for finding the $(n+1)$th element of the sequence given the nth element [5]. Now we see that this very closely models the method of proof by mathematical induction, except in this case used only as a method of definition rather than proof. With this basic notion of recursive definition in mind, we can then define an arithmetical *function* as being recursive if it is the last term in a finite sequence of functions, with each such function being recursively defined using the two previous functions in the sequence [5]. A recursive function may also be "obtained by substitution from a preceding function" or the successor function of some constant [5, p. 12]. Thus the definition of a recursive arithmetical function corresponds more closely to the strong, or second, principle of mathematical induction than to the first principle. Lastly, the recursiveness of any other concept in arithmetic is defined using recursive functions [5]. As we will see, Gödel makes almost exclusive use of recursive definitions and concepts in his paper.

The importance of recursiveness is largely in the fact that it allows for primarily constructive arguments to be used. A *constructive* argument is one that involves the careful creation of certain objects according to one or more given rules. This type of proof is valued because it is "intuitionistically unobjectionable," since all assumptions, particularly existential ones, are rather clearly justified [4, p. 60]. Rather than using a particular tricky step or reasoning that does not seem wholly justified, a constructive argument focuses on building the desired situation piece by piece in such a way that each step is reasonable. Hence in a properly constructive proof, no one can reasonably object to the conclusion without rejecting some part of the construction. Additionally, in this case remarks about a recursively defined infinite sequence can be interpreted as remarks about the rule of construction itself. This is yet another reason that recursive definitions lend themselves well to constructive arguments [5]. These benefits of recursiveness apply to most metamathematical problems and not only to Gödel's work.

Recursive definitions are used by Gödel effectively because of an important proposition whose proof he outlines in his paper. This proposition states essentially that any recursive definition or relation can be expressed as a formula f within Gödel's formal system P. This formula f is constructed in such a way that if the relation is true, then f is provable in P; if the relation is false, then $\neg f$ is provable in P [5]. This result allows Gödel to utilize recursive definitions efficiently within his formal system. Additionally, it enables such recursive relations to be discussed in terms of provability and unprovability within the language of the system P.

The Theorems and Proofs

With the major machinery of the proofs in place, we can begin to discuss Gödel's main arguments in his paper, though we will avoid getting into details for the sake of brevity. The crux of Gödel's argument for the First Incompleteness Theorem is his idea of constructing, within his formal system P, a specific proposition in whole number arithmetic that cannot be proven or disproven. The specific statement he creates closely mirrors that of the famous Liar's Paradox in logic, which in its simplest form is the following claim: "This statement is false" [7]. If assumed to be true, then it must be false, providing a contradiction. If assumed false, it must then be true, providing yet another contradiction. Similarly, the statement Gödel constructs is essentially this: "This statement is unprovable." This sentence is then provable and so is its negation, meaning that the system it is in cannot be consistent. Hence, if the system is *assumed* consistent, the statement is undecidable [4]. The argument for the Second Incompleteness Theorem is built on previous results in the paper and is similar to the first in the sense that Gödel constructs a sentence asserting the consistency of the formal system. This sentence cannot be provable, for if it were then the statement that was constructed and shown to be undecidable in the first theorem would then also be decidable. Thus if the system is consistent, it cannot prove its own consistency [4].

Gödel begins his paper by specifying the formal system that he intends to use to prove nearly every result in his paper. His formal system *P* essentially consists of the part of the system within *Principia Mathematica*, the colossal work by Bertrand Russel and Alfred North Whitehead from 1910-1913, necessary to establish the full theory of arithmetic [5]. Gödel then identifies the symbols used to create formulas within his formal system *P* and defines several key terms. He then describes the axioms utilized within *P*. He also describes the two rules of inference to be used within *P*: the conditions of being an "immediate consequence" of two formulas and of one formula, respectively [4, p. 45]. Now with the core description of *P* completed, Gödel begins describing his method of mapping formulas within *P* into numerical values. Often called Gödel numbering, this is the same method involving primes as discussed previously.

At this point, Gödel shifts his focus and begins discussing recursion. First, he gives his definition of a function (or relation) that is recursively defined. Then he presents four propositions concerning recursive functions and their various properties, proceeding to outline proofs of each of these propositions. With these propositions in mind, he continues by recursively defining forty-five relations that correspond to different properties of the natural numbers. For instance, Prim(x) means "x is a prime number" and corresponds to a particular formula [4]. These recursive functions provide the main way in which relations between formulas in P are expressed as relations between corresponding Gödel numbers.

Now we come to the primary result of this paper, what Gödel calls "the object of our exercises" [4, p. 56]. This major result is referred to as simply "Proposition VI" and can be thought of as a more general version of the First Incompleteness Theorem; the first theorem does not appear in its widely-known form until later. We now state this major proposition in its entirety and attempt to clarify unfamiliar terminology.

Proposition VI *To every ω-consistent recursive class c of formulas there correspond recursive class-signs r, such that neither v Gen r nor ¬(v Gen r) belongs to Flg(c) (where v is the free variable of r)* [4, p. 57].

First, c is simply defined to be a general "class" or collection of recursively defined formulas. The notation Flg(c) represents ". . . the smallest set of formulas which contains all the formulas of c, all axioms, and is closed with respect to the relation 'immediate consequence of'" [4, p. 57]. Essentially, this is the set of all formulas that can be proven by axioms of the system and formulas in c, with the set containing the axioms as well as the formulas already in c. What is meant by ω-consistency is essentially a stronger form of the commonly understood consistency. Thus ω-consistency implies standard consistency, however the converse is not true [4]. It was discovered later that only the assumption of standard consistency is necessary for this first theorem to apply, however Gödel uses this stronger assumption to simplify the details of the proof somewhat. The term "recursive class-sign" refers to a specific type of recursively defined formula expressible within *P*. Finally, v Gen r and $\neg(v$ Gen $r)$ simply represent the specific formulas Gödel shows to be unprovable. Note that x Gen y is one of the forty-five recursive definitions given earlier by Gödel for variables x and y. Also, one should note that the statement "neither v Gen r nor $\neg(v$ Gen $r)$ belongs to Flg(c)" essentially means that v Gen r and its negation are both not in the class of consequences of the axioms and any formulas in c. Ultimately, this is the statement of unprovability, for if v Gen r or its negation were provable using the axioms and formulas in c, then one or the other would belong to Flg(c). The specific formula that is shown to be unprovable can be formulated as 17 Gen r.

After delivering a very careful and detailed proof for such a strong result, Gödel begins to apply it more specifically to the case of arithmetical propositions. He first defines precisely what it means for a relation to be "arithmetical," and follows by showing that every recursive relation is arithmetical [4]. Further, this result is able to be formalized within *P*, with the same equivalence between recursive relations and arithmetical relations being expressed in *P* as well [4]. Now applying this result along with that from Proposition VI, we reach what is most widely known as the First Incompleteness Theorem.

Proposition VIII *In every one of the formal systems referred to in Proposition VI there are undecidable arithmetical propositions* [4, p. 65].

Note that the formal systems this explicitly applies to are those that are ω-consistent and are obtained from *P* by adding a certain amount of recursively defined axioms (i.e. the formulas in c).

Finally we come to the last main result in Gödel's paper, usually referred to as the Second Incompleteness Theorem.

Proposition XI *If c be a given recursive, consistent class of formulas, then the propositional formula which states that c is consistent is not c-provable; in particular, the consistency of P is unprovable in P, it being assumed that P is consistent* [4, p. 70].

Here we see that the class *c* of recursively defined formulas can be understood in the same way as in Proposition VI, though it may not necessarily be the exact same class. Also, a statement being *c*-provable means that it is provable using the formulas from *c* as well as the basic axioms of the formal system.

Rather than discussing details, Gödel provided a sketch of the proof of this final proposition. The essence of his argument in this case is to specify a formula claiming that *c* is consistent within *P*. Then if this formula is provable, the formula 17 Gen *r* must also be provable. However, this was the formula specifically shown to be undecidable in Proposition VI, and so *c* must be inconsistent within *P* [4]. While only an outline, the argument does seem reasonable enough in light of Gödel's previous results.

Gödel seemingly intended to write a "sequel" to his paper. Its purpose would be to generalize the results more explicitly to other formal systems. This is because in their original form they apply only to the system *P*, though he does briefly indicate how they might apply to other systems [4]. He also intended to prove Proposition XI in full detail. However, this "sequel" was never published. Nevertheless, other mathematicians have filled in the details of Proposition XI, showing that it is indeed valid, as well as carrying out generalizations to other formal systems [8].

Immediate Consequences

Having discussed the results of Gödel's paper, it will be useful now to clarify precisely what his results mean and where they are truly applicable. Gödel's First Incompleteness Theorem is ultimately a statement about formal systems capable of expressing a certain amount of arithmetic [4]. Thus, if a given formal system is incapable of containing the necessary arithmetic, then the First Incompleteness Theorem does not apply. However we also note here that the first theorem applies to systems with objects that can be expressed as numbers in a one-to-one correspondence [8]. For example, the first theorem applies to the foundational system of set theory (ZFC) even though no explicit numerals are given as axioms; the natural numbers are recursively defined based on the empty set. Furthermore, in reference to the existence of undecidable statements, the first theorem only states that in a formal system satisfying the hypotheses there exists an *arithmetical* proposition which is undecidable within the system [8]. Hence in a formal system to which the first theorem would apply, if that system describes more than just number theoretic concepts, the first theorem does not necessarily state that there exist undecidable propositions regarding anything other than arithmetic. Similarly to the earlier caveat, it is conceivable that the first theorem does describe the existence of an undecidable proposition that is equivalent to some arithmetical claim. Finally, while this will not affect interpretations as much, it is important to note that Gödel's first theorem is only applicable if the formal system under consideration is assumed consistent [4]. Without this assumption, the theorem does not apply, though it is usually not an unreasonable assumption to make.

While Gödel's First Incompleteness Theorem is described within his paper only in terms of his formal system *P*, Gödel himself indicates that this is not the only system this result applies to. After proving the first theorem, he continues by describing only two conditions that a formal system must satisfy in order for the first theorem to apply [4]. He did not provide an explicit proof of this and one was not provided in any form until later on, however this is an essential comment to make. If this first theorem only applied to Gödel's system *P* and perhaps a few other specific cases, it would not be nearly as notable as it is. In reality, the first theorem is applicable to a considerably large class of formal systems.

Gödel's Second Incompleteness Theorem does seem to be more straightforward regarding assumptions and interpretations, however there are still some important things to note. First, this result also holds only for formal systems capable of expressing a certain amount of arithmetic. Interestingly, this "certain amount" of arithmetic is somewhat different from what is required in the first theorem [8]. Additionally, this result again requires that the formal system be assumed consistent. If it is not, then, as Gödel himself writes, ". . . every statement is provable" [4, p. 70]. Once again this is a reasonable assumption, but one should be aware of it regardless.

Truth and Doubt in Mathematics

Another perceived impact of the Incompleteness Theorems is the claim that they may shed doubt upon the consistency of mathematics as well as its ability to determine truth. Many discussions of this variety are often exaggerated, although the origination of these concerns may seem wholly reasonable. First is the fact that Gödel's first theorem claims that there exist arithmetical statements which are *true* but unprovable in the given system. If these propositions are known to be true, then what does that imply about modern systems that this theorem applies to? Does this mean that current deductive systems are insufficient to express unquestionably true statements and that perhaps a different system is needed to adequately express truth? Another reason for these concerns is Gödel's second theorem, stating that no applicable formal system can prove its own consistency. Does this mean that modern systems are inconsistent and that their results should be called into question? These questions can be answered with a thorough understanding of the limitations of Gödel's theorems as well as some light philosophical discussion.

The First Incompleteness Theorem states in part that there exist true but undecidable propositions within an applicable, consistent system. However, what is meant by the word "true" is *not* that the propositions represent some absolute truth that is known. Rather in a mathematical sense, propositions are either true or false. If a proposition p is undecidable, then neither p nor $\neg p$ can be proven. However one of these statements is mathematically true; for if p is false, then $\neg p$ by definition is true. Additionally, rather than representing an absolute truth, an undecidable proposition q is true in the sense that the statement "If P is consistent, then q is true" is a true statement for a formal system P [8]. If q is true, then by Gödel's mapping there must be some arithmetical property corresponding to q that is true as well [3]. This is ultimately where the notion of a true but unprovable statement in arithmetic comes from. However this inherent incompleteness should not mean that no arithmetical system is adequate. What is adequate for a mathematical system is, generally, what is most interesting and useful. As Ernst Nagel and James R. Newman right, ". . . it gradually became clear that the proper business of the pure mathematician is to *derive theorems from postulated assumptions*, and that it is not his concern as a mathematician to decide whether the axioms he assumes are actually true" [3, p. 7]. Thus there is no reason based solely on the first theorem to believe that current systems are insufficient.

Gödel's Second Incompleteness Theorem states that a formal system satisfying the necessary assumptions cannot prove its own consistency. However this should not be sufficient to cast doubt on all deductive systems. Generally, one is only concerned with proving the consistency of a system when there is genuine concern the system may not be consistent. If such a proof of consistency within the system were to be produced, there would be no reason to take this as reassurance that the system is consistent. This is because such a proof uses axioms that are already under scrutiny, and thus would still be treated with a degree of uncertainty [8]. Recall that anything can be proven in an inconsistent system. Thus if the formal system under consideration is truly inconsistent, then it would have no problem proving its own consistency as well as its own inconsistency. With this in mind, the fact that absolute proofs of consistency are generally impossible should not lead to despair or extreme doubt in mathematics. What is possible, and has been demonstrated numerous times throughout history, is assuming the consistency of another system that is *not* in doubt then using that assumption to prove the consistency of the system that *is* in doubt [8]. This is precisely what was accomplished in the early nineteenth century regarding the relative consistency of hyperbolic geometry to Euclidean geometry.

Invocations in Other Disciplines

Often the Incompleteness Theorems are referenced in discussions in disciplines other than mathematics when they should not be or when they are simply unnecessary to the argument being made. For example, a student of law may attempt to argue that the constitution of a certain nation is either incomplete or inconsistent. If this student appeals to Gödel's theorems to make his or her argument, then he or she is misinterpreting critical aspects of the theorems as well as introducing an extraneous argument [8]. Recall that Gödel's theorems apply only to formal systems in which a certain amount of arithmetic is able to be formalized. It is rather safe to say that no country's constitution fits the stringent, mathematical requirements necessary to be considered a formal system [8]. Constitutions usually do not contain any axioms assumed to be true, they contain no rules of logical inference that must be clearly followed, and they prove no theorems. In addition to this, a constitution that would be capable of expressing arithmetic would certainly be a rarity. Thus the theorems cannot correctly be applied to such a situation.

Another aspect of this hypothetical that deserves discussing is that the Incompleteness Theorems are not necessary to the student's arguments about incompleteness and inconsistency. First, it should be intuitively clear that any document is inherently incomplete in the sense that it cannot possibly address every possible circumstance [8]. New situations are constantly arising that call for discussion and reinterpretation of legalities; in fact, this is the entire purpose of most judicial systems. If a legal document were complete, there would theoretically be no need for judges to interpret laws, as each specific situation would have been addressed within the document itself. Turning to the inconsistency portion of the argument, it is very possible for legal documents and systems to be inconsistent, particularly if they continue to grow in size to attempt to address new situations. With such a large body of legal code managed and updated by errant humans, it is almost inevitable that inconsistencies will result [8]. Again, part of the responsibility of the legislative body as well as the judicial body is to manage this and resolve inconsistencies. Finally, we note that such a student's argument that a nation's legal code is either incomplete or inconsistent is actually something of a false dichotomy. Based on our arguments, it is perfectly reasonable to envision a collection of laws that is both incomplete and inconsistent.

Many misapplications of Gödel's theorems can be critiqued in a similar way as discussed here in the hypothetical. On occasion, authors aiming to make theological or philosophical arguments make appeals to the Incompleteness Theorem [8]. However the key difference between these disciplines and formal systems in mathematics is essentially the same as with the law hypothetical. As Torkel Franzén states in his critique of applications of incompleteness, "Whether something follows from what is said in the Bible is not a mathematical question, but a question of judgment, interpretation, belief, opinion" [8, p. 78]. The logic of arguments in theology and philosophy are often up for debate with reasonable arguments on most sides, whereas mathematical conclusions are *not* up for debate, so long as they adhere to the necessary rules of inference and assumption. This is yet another way in which mathematics differs from many other disciplines attempting to apply Gödel's theorems.

Human Capacity and Thought

Frequently Gödel's theorems are invoked in efforts to discuss the limitations of the human mind in regards to logic and proof. It has been claimed that the human mind is in some sense a formal system and thus is subject to the constraints of the Incompleteness Theorems. However, the same objections raised previously may apply to this discussion: there are no clearly defined axioms, rules of inference, etc. governing the human mind [8]. Thus we have reasonable grounds to argue that Gödel's theorems do not explicitly apply to human thought. If we conclude that human thought is not constrained by the theorems, must it then follow that there is a certain irreducible aspect of the human mind that transcends traditional machinery or programming [8]? One reason for this argument is that computing machines are programmed to follow certain rules and procedures. Thus if they are advanced enough, they may model formal systems accurately. Many mathematicians would disagree with this conclusion while many others would agree and argue in favor of it. Some of these perspectives will be discussed to gain a better understanding of the issue.

A major goal of some mathematicians and philosophers is to refute the concept known as mechanism, which is the idea that human minds can be fully expressed as machines [9]. Often the Incompleteness Theorems are used as a starting point for these arguments. To summarize the state of such an argument, by following the logic of Gödel's first theorem, the human mind should be able to prove that the Gödel sentence of a formal system is in fact true, where the Gödel sentence of a system is that statement which asserts its own unprovability. However, this sentence is only true if the consistency of the system is assumed; thus the logician may argue that the human mind is capable of confirming the consistency of any formal system [9]. This assumption is rather too strong to accept in most cases. However, it became clear that *some* assumption was necessary for any hope of convincingly refuting mechanism [9]. Eventually some more reasonable assumptions were identified that would have provided a convincing proof if not for the existence of a paradox within the assumptions [9]. Hence logicians began searching for ways to resolve this paradox. More recent work has shown that if a particular resolution of this paradox is accepted, then mechanism is ultimately shown to be false [9]. Thus, much progress has been made in support of antimechanist arguments, however the work under consideration is still open to criticism, as its proof requires many additional assumptions and thus leads to a weaker result overall.

On the other side, many prominent mathematicians reject the notion that human thinking transcends the limitations of a machine. In some of these situations, the basic distinction between discussing a theory and discussing a metatheory in logic is very important [10]. Recall that a metatheory is simply any body of knowledge or discussion about the structure of the originial theory itself. Neglecting this distinction, however, may lead to problems for proponents of the transcendence of the human mind. As one author writes, “A person reasoning about a machine knows the machine completely, thus there is nothing surprising in being able to produce something that the machine cannot prove” [10, p. 336]. However when humans attempt to reason about themselves, the same thing cannot be said, for no human knows his own mind completely [10]. This calls into question whether humans can “prove” anything about their own minds, especially considering the vast number of mysteries surrounding the mind that remain.

To some logicians, what is more relevant to this discussion is determining what mathematical statements one is willing to accept as intuitively true [10]. This question has been considered for many years and was central to Hilbert in particular. Pudlák argues in his article that it is something of a misconception that a system can always add true, independent statements [10]. In a sense, he means that mathematical statements often become less intuitively true as the assumptions required to express it grow stronger. Hence his ultimate claim is that, “If we always justify a weaker principle by a stronger one, we are inevitably [led] to so strong principles that their truth is not evident to us” [10, p. 341]. As an illustration, in many cases basic axioms are chosen for their self-evident truth. Statements that are independent of a system with these axioms may be self-evident for some, but not so much to others, as was the case with Euclid's parallel postulate in the *Elements*. If such a statement is added to the axioms to create an extended system, it would stand to reason that propositions independent of this new system would be less evidently true than previous independent propositions, and thus the cycle continues.

Ultimately it is difficult to accept conclusively a specific perspective in the discussion on the essentially transcendent nature of the human mind. All views seem to have convincing arguments, albeit some to different degrees. A point that further complicates matters is that it is near-impossible to describe what the human mind can "prove" in mathematics [8]. Throughout the years there have been such a variety of changing views on mathematics, with some results being initially rejected but clearly understood and accepted by later generations. Clearly, intuition in math changes as new results are reviewed and extended. As one is exposed to more advanced topics in mathematics, intuition can often work against the student, and even the professional. However as new results become accepted, students and professionals are able to develop the correct intuition over time. In addition to this, an incredibly diverse set of factors affect the workings of the human mind, from genetics and environment to opportunity, attitude, and motivation. There is simply not enough information about the mind to allow one to reasonably discuss what humans are capable of proving [8]. Gödel and his ingenious methods of proof are a prime example of the incredible creativity of the human mind in regard to logic.

The Impact of Gödel's Results

Gödel's theorems have had a rather large impact on mathematics and the world at large, however it seems as though this impact is often overstated and improperly applied in popular society and in other major disciplines. It is often claimed that these results revolutionized the study of mathematics at large as well as that of physics and other related fields. In some sense this is true, however this does not mean that these theorems get used regularly by mathematicians of various fields. Their impact is more that of an interesting and unexpected result that wounded some popular conceptions of the early twentieth century while not directly affecting the studies of most working mathematicians. The theorems can also inspire awe at the creativity of the human mind as it pertains to logic and reasoning. Even though most mathematicians are not directly impacted by these theorems on a regular basis, they still provide fascinating results and are the foundation for often interesting discussions.

As has been discussed thus far, Gödel's result applies mainly to arithmetical results, which certainly makes one wonder what is so important about them. The study of arithmetic is one of the most ancient in mathematics and is central to the understanding of countless other fields within the discipline. The fact that there are propositions of such a fundamental subject that cannot be decided is certainly a shocking consequence of the theorems. Another reason these results are so important is the novel techniques used to prove the theorems and concepts related to it. For example, Gödel's mapping of metamathematical statements about a system into the language of the system itself was a rather unique idea at the time. The notions of recursive definitions and recursive functions have also become increasingly important since Gödel's work in which they played a central role. Finally, Gödel's method of proof was described as being "deeper" and "more subtle" than any previous metamathematical proof, which certainly contributed to its popularity [5].

Conclusion

The discussions surrounding independent statements and consistency have evolved rapidly over the past century. Far from being a settled topic, these issues continue to be addressed from different perspectives, although certainly much more is known about them now than previously. It seems as though, as a result of Gödel's Incompleteness Theorems, these topics will continue to be of interest for as long as mathematics itself is of interest. For according to the theorems, there will always exist independent arithmetical statements within applicable systems, hence more undecidable statements to consider. In addition, while no system can prove its own consistency, this does not require abandoning the idea of consistency as a whole; there are many different approaches that continue to be explored, with relative consistency most prominent among them.

Gödel's theorems are remarkable not only for the results presented in them, but also for the methods used, several of which were pioneered by Gödel himself. His numbering system and subsequent technique for expressing meta-arithmetical statements within arithmetic itself is certainly a stroke of genius. Many logicians also realized the potential of recursive definitions in metamathematics, hence these have also been widely used since Gödel. This in combination with an argument resembling the Liar's Paradox leads to surprising but fascinating results.

As important as they are to logic and mathematics, the Incompleteness Theorems are very prone to misapplication in many disciplines outside mathematics. Often these misuses are characterized by a misunderstanding of the assumptions necessary for the theorems to apply. Usually this results in them being invoked outside of formal systems capable of expressing arithmetic, and sometimes even outside of any formal system whatsoever. Their use in these situations is also usually superfluous, as it seems intuitively obvious that most human constructs outside of mathematics will at minimum be incomplete, and perhaps inconsistent as well depending on the construct. While not directly applying, the theorems can serve as an inspiration for many other discussions or approaches, for example the debates on the transcendental nature of the human mind. Often such inspiration leads into philosophy more than mathematics.

The Incompleteness Theorems do not invalidate any major established results in mathematics nor do they inherently limit what is mathematically possible. In reality, they have helped mathematicians to come to a more accurate and proper understanding of the work they do and the purpose of it. No matter one's particular interpretation, these theorems should not be a source of doubt or even despair in mathematics. On the contrary, they should inspire wonder at the creativity of the human mind and curiosity as to what advances may follow.

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