# Examples of Solving the Wave Equation in the Hyperbolic Plane

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A Senior Thesis submitted in partial fulfillment
of the requirements for graduation
in the Honors Program
Liberty University
Spring 2018

Acceptance of Senior Honors Thesis

This Senior Honors Thesis is accepted in partial fulfillment of the requirements for graduation from the Honors Program of Liberty University.

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Abstract

The complex numbers have proven themselves immensely useful in physics, mathematics, and engineering. One useful tool of the complex numbers is the method of conformal mapping which is used to solve various problems in physics and engineering that involved Laplace's equation. Following the work done by Dr. James Cook, the complex numbers are replaced with associative real algebras. This paper focuses on another algebra, the hyperbolic numbers. A solution method like conformal mapping is developed with solutions to the one-dimensional wave equation. Applications of this solution method revolve around engineering and physics problems involving the propagation of waves. To conclude, a series of examples and transformations are given to demonstrate the solution method.

*Keywords:* hyperbolic, Fourier, associative real algebra, wave equation

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## Introduction

Conformal mapping refers to local angle preserving transformation functions called conformal maps. These maps turn out to be profoundly useful in physics when looking at them in the complex number plane. Certain transformation functions in the complex plane are solutions to Laplace's equation which is the equation governing the temperature distribution in two-dimensional steady-state heat flow and electrostatics and a variety of other physical phenomena (Saff & Snider, 2016). The solutions to many problems in physics are found in the study of various conformal maps in the complex numbers. Similar mappings can also be studied on the hyperbolic number system which houses functions that are solutions to the wave equation. The study of these hyperbolic mappings is in its infancy and, as a result, its applications to physics are slim; but considering how useful complex conformal mapping has been to physics, it is reasonable to suppose that similar benefits may be derived from the hyperbolic numbers. To begin, a study of complex conformal mapping and its applications to physics will be undertaken. Following this, the theory of associative real algebras will be studied using Dr. James Cook's paper *Introduction to A-Calculus* (2017) as a guide. The larger picture that encapsulates both the complex and hyperbolic number systems will be highlighted as well. Having created a framework for studying associative real algebras, the focus will shift to the hyperbolic numbers which are an instance of associative real algebras. Using the developed theory, the hyperbolic number system will be studied to discover useful transformations that solve the wave equation. A series of example transformations and regions will be presented to conclude the paper.

## Complex Conformal Mapping

A common technique for solving Laplace's equation is the conformal mapping technique found in the study of the complex numbers. Laplace's equation can be solved on a variety of regions with this method. To establish the conformal mapping technique, the complex numbers must first be examined and understood. The theorems developed here will be utilized as a framework for studying the hyperbolic numbers later in the paper.

### The Complex Numbers

This section follows *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics* by Edward B. Saff and Arthur D. Snider (2016). This section is not a complete study of complex analysis and should not be treated as such. Rather, only the necessary theorems and definitions for studying the conformal mapping technique will be covered. For complete works on complex analysis, the works by Brown & Churchill (2009), and Saff & Snider (2016) are excellent references.

Definition 1. A complex number is an expression of the form $a + bi$, where $a$ and $b$ are real numbers. Two complex numbers $a + bi$ and $c + di$ are said to be equal if and only if $a = c$ and $b = d$.

For complex number $a + bi$, $a$ is called the real part and denoted by $\mathrm{Re}(a + bi) = a$, and $b$ is the imaginary part denoted by $\mathrm{Im}(a + bi) = b$.

Definition 2**.** The modulus of the number $z = a + bi$, denoted $|z|$, is given by

$$|z| = \sqrt{a^2 + b^2}. \tag{1}$$

Definition 3. The complex conjugate of the number $z = a + bi$ is denoted by $\bar{z}$ and is given by,

$$\bar{z} = a - bi. \tag{2}$$

Another immensely useful formula in the complex numbers is Euler's Equation.

Definition 4. If $z = x + iy$, then $e^z$ is defined to be the complex number,

$$e^z = e^x(\cos y + i \sin y). \tag{3}$$

**Analytic Functions**

Before discussing analytic functions in detail, first, the definition of a circular neighborhood is needed.

Definition 5. The set of all points that satisfy the inequality,

$$|z - z_0| < \rho, \tag{4}$$

where $\rho$ is a positive real number and $z, z_0$ are complex numbers, is called an open disk or circular neighborhood of $z_0$.

Having established the definition for a circular neighborhood, the definition of the limit of a complex-valued function can be made.

Definition 6. Let $f$ be a function defined in some circular neighborhood of $z_0$, with the possible exception of the point $z_0$ itself. We say that the limit of $f(z)$ as $z$ approaches $z_0$ is the number $w_0$ and write,

$$\lim_{z \to z_0} f(z) = w_0 \tag{5}$$

or equivalently,

$$f(z) \to w_0 \quad as \quad z \to z_0 \tag{6}$$

if for any $\epsilon > 0$ there exists a positive number $\delta$ such that,

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{7}$$

From Definition 6, the definition of continuous complex-valued functions is natural.

Definition 7. Let $f$ be a function defined in a circular neighborhood of $z_0$. Then $f$ is continuous at $z_0$ if,

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{8}$$

A function $f$ is said to be continuous on a set $S$ if it is continuous at each point in $S$. The observant reader may have noticed that both definitions are closely related to their real number counterparts. This trend continues, and all the usual limit properties seen in calculus over the real numbers are found here. The real counterparts to each of the above definitions and below properties can be found in the calculus textbooks by Salas, Hille & Etgen (2007), and Briggs, Cochran, Gillett, & Schulz (2013).

Theorem 1. If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then,

1. $\lim_{z \to z_0} \big(f(z) \pm g(z)\big) = A \pm B,$
2. $\lim_{z \to z_0} f(z)g(z) = AB,$
3. $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ if $B \neq 0$.

Proof: Proof of the previous theorem can be found on pages 61-62, of Saff & Snider (2016). ■

The continuity properties from the real numbers also translate to the complex numbers.

Definition 8. Let $f$ be a complex-valued function defined in a circular neighborhood of $z_0$. Then the derivative of $f$ at $z_0$ is given by,

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{9}$$

provided this limit exists.

Any complex-valued function $f$ satisfying the above definition is said to be complex-differentiable at $z_0$. Again, all the usual derivative properties hold for the complex numbers.

Theorem 2. If $f$ and $g$ are differentiable at $z$, then,

1. $(f \pm g)'(z) = f'(z) \pm g'(z)$,
2. $(cf)' = cf'(z)$ (for constant $c$),
3. $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$,
4. $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$.

If $g$ is differentiable at $z$ and $f$ is differentiable at $g(z)$, then the chain rule holds:

$$\frac{d}{dz} f\big(g(z)\big) = f'\big(g(z)\big)g'(z). \tag{10}$$

Proof: This theorem is proved in detail in on pages 68-68, of Saff & Snider (2016). ■

Any complex-valued function $f(z)$ can be written in terms of its real and imaginary component functions $u(x, y)$ and $v(x, y)$. Thus, $f$ can be written $f(z) = u(x, y) + iv(x, y)$.

This leads to an important property that many complex-valued functions have.

Definition 9. A complex-valued function $f(z)$ is said to be analytic on an open set $G$ if it has a derivative at every point of $G$.

Analytic functions are covered later in more detail, but first the Cauchy-Riemann equations and the theorems they lead to are needed.

**Laplace's Equation**

Using the limit properties from the previous section paired with the structure of complex numbers, the derivative of a complex-valued function at $z_0$ is shown to be,

$$f'(z_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{11}$$

It is also the case that,

$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0). \tag{12}$$

Thus, by equating the real and imaginary parts, the below equations are obtained:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{13}$$

The equations above, Equation 13, are the famed Cauchy-Riemann equations. This result provides a necessary condition for a function to be complex-differentiable, i.e. analytic.

Theorem 3. If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point, $z_0 = x_0 + iy_0$ then the Cauchy-Riemann equations must hold at $z_0$. As a result, if $f$ is analytic in an open set $G$, then the Cauchy-Riemann equations must hold at every point in $G$.

Proof: A detailed proof of this theorem can be found on pages 73-74 of Saff & Snider (2016). ■

Theorem 3 is useful, but notice that the Cauchy-Riemann equations alone are not a sufficient condition for complex-differentiability. To achieve this, the first partial derivatives of $u$ and $v$ must be continuous.

Theorem 4. Let $f(z) = u(x, y) + iv(x, y)$ be defined on some open set $G$ containing a point $z_0$. If the first partial derivatives of $u$ and $v$ exist in $G$, are continuous at $z_0$, and satisfy the Cauchy-Riemann equations at $z_0$, then $f$ is complex-differentiable at $z_0$.

Proof: Proof for this theorem can be found on pages 74-76 of Saff and Snider (2016). ■

Following James Brown & Ruel Churchill (2009), Saff & Snider's theorem can be added to by providing formulas to simplify the calculation of derivatives of complex-valued functions,

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{14}$$

Repackaged with more elegant notation, the formula reads,

$$f'(z_0) = u_x + iv_x. \tag{15}$$

Where $u_x = \frac{\partial u}{\partial x}$ and $v_x = \frac{\partial v}{\partial x}$. This formula can also be derived for component function $v$.

This leads to the concept of harmonic functions. These functions wrap up continuity with the two-dimensional Laplace equation and lead to a fantastic result of analytic functions. Recall that Laplace's equation is written,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{16}$$

For further study in Laplace's equation and its applications to physics, see Gustafson (1980), Haberman (1998), Logan (2015), and Weinberger (1995).

Definition 10. A real-valued function $\phi(x, y)$ is said to be harmonic in a domain $D$ if all its second-order partial derivatives are continuous in $D$ and if, at each point in $D$, $\phi$ satisfies Laplace's equation, Equation 16.

This definition leads to the following theorem.

Theorem 5. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$, then each of the functions $u(x, y)$ and $v(x, y)$ is harmonic in $D$.

Proof: To complete this proof the real and imaginary parts of any analytic function must have continuous partial derivatives of all orders. This result is not covered in this paper, but Saff & Snider (2016) prove this in detail. This result is assumed moving forward. From calculus it is known that partial derivatives commute under the conditions on the functions so,

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y}. \tag{17}$$

Apply Equation 13 to Equation 17 and obtain,

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}. \tag{18}$$

Thus $v$ is harmonic. A similar argument proves $u$ is harmonic. ■

With Theorem 5, solutions to Laplace's equation are easily found. Harmonic functions solve Laplace's equation, so any analytic function automatically has its component functions solve Laplace's equation.

**Conformal Mapping**

Conformal mapping is the study of the geometric properties of functions considered as mappings from a domain to a range.

Theorem 6. If $f$ is analytic at $z_0$ and $f'(z_0) \neq 0$, then there is an open disk $D$ centered at $z_0$ such that $f$ is one-to-one on $D$.

Proof: The proof of this theorem requires the study of integration of complex functions which is beyond the scope of this paper, so a detailed proof can be found on page 378, of Saff & Snider (2016). ■

Now the definition of conformality can be given. To understand the coming definition, consider an analytic, one-to-one function $f(z)$ in a neighborhood of the point $z_0$. Also, consider two smooth curves $\phi_1$ and $\phi_2$ intersecting at $z_0$. Under the mapping $f$, the images of these smooth curves $\overline{\phi}_1$ and $\overline{\phi}_2$ are also smooth curves intersecting at $f(z_0) = w_0$. Construct vectors $v_1$ and $v_2$ at $z_0$ that are tangent to $\phi_1$ and $\phi_2$ respectively that are pointing in the direction of the orientation of the two curves. The angle from $\phi_1$ to $\phi_2$, $\theta$, is the angle through which $v_1$ must be rotated counterclockwise to lie along $v_2$. Define the angle $\theta'$ from $\overline{\phi}_1$ to $\overline{\phi}_2$ similarly. The mapping $f$ is conformal at $z_0$ if $\theta = \theta'$ for every pair of smooth curves intersecting at $z_0$. This condition is referred to by saying that the angles are preserved (Saff & Snider, 2016). When $f$ is analytic, Theorem 7 follows.

Theorem 7. An analytic function $f$ is conformal at every point $z_0$ for which $f'(z_0) \neq 0$.

Proof: A full proof of this theorem is beyond the scope of this paper, but a proof sketch is given. By Theorem 7, there is an open disk containing the point $z_0$ where $f$ is one-to-one. Every smooth curve through $z_0$ has its tangent line through the same angle, under the mapping $w = f(z)$. Thus, the angle between any two curves intersecting at $z_0$ will be preserved. A full proof can be found on pages 378-379 of Saff & Snider (2016). ■

The problem of mapping one domain into another under a conformal transformation can be difficult, so developing the basic building blocks that allow the study of a large range of conformal mapping problems is proper. These tools are the properties of Möbius Transformations which will be discussed in the next section.

Consider first the translation mapping defined by the function

$$w = f(z) = z + c \tag{19}$$

for a fixed complex number $c$. This mapping shifts every point by the vector $c$. Clearly, the angles here are preserved.

The rotation mapping utilizes Euler's equation and is written

$$w = f(z) = e^{i\theta}z. \tag{20}$$

This transformation rotates each point about the origin by angle $\theta$. Again, this preserves the angles between the domains.

The magnification transformation is written

$$w = f(z) = pz \tag{21}$$

where $p$ is a positive real constant. This transformation simply inflates the original domain by a factor of $p$. This leaves the angles unchanged, thus preserving them.

A linear transformation is a mapping of the form,

$$w = f(z) = az + b \tag{22}$$

for complex constants $a$ and $b$ with $a \neq 0$. These are important because we can view them as the composition of a rotation, a magnification, and a translation. The final building block transformation is the inversion transformation written,

$$w = f(z) = \frac{1}{z}. \tag{23}$$

This transformation has the fantastic property that its image of a line or circle is always either a line or a circle. The details of this property are outlined on pages 388-389, of Saff & Snider (2016).

**Möbius transformations.**

Definition 11. A Möbius Transformation is any function of the form,

$$w = f(z) = \frac{az + b}{cz + d} \tag{24}$$

with the restriction that $ad \neq bc$. (This condition excludes $w$ being a constant function.)

Any Möbius transformation can be decomposed into a composition of the previously described building block transformations. This statement is formalized in Theorem 8.

Theorem 8. Let $f$ be any Möbius transformation. Then,

1. $f$ can be expressed as the composition of a finite sequence of translations, magnifications, rotations, and inversions,
2. $f$ maps circles and lines to themselves,
3. $f$ is conformal at every point except the poles.

Proof: First notice that,

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \tag{25}$$

is only equal to zero at $z = -d/c$ and thus is conformal at every point other than that. Now consider the linear transformation induced by $c = 0$. For $c \neq 0$, write,

$$\frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}. \tag{26}$$

This shows that the Möbius transformation can be expressed as a linear transformation,

$$w_1 = cz + d \tag{27}$$

followed by an inversion,

$$w_2 = \frac{1}{w_1} \tag{28}$$

and then another linear transformation,

$$w = \left(b - \frac{ad}{c}\right) w_2 + \frac{a}{c}. \tag{29}$$

The proof for (8.2) was outlined previously by referencing Saff & Snider pages 388-389 (2016). ■

Having established the basic tools of conformal mapping, the solution technique can now be studied. Knowing that any analytic function has solutions to Laplace's equation built into its component functions, there are an innumerable number of solutions that can be easily found for different regions in the complex plane. For example, it can be shown that the level curves of the logarithm function form a circle in the complex plane. It can also be shown that the logarithm function is analytic, and thus its component functions are solutions to Laplace's equation. Using this fact, transformation functions can be found whose level curves have images that will be a circle, a region with a known solution to Laplace's equation. With this transformation, the composition of the transformation function with the solution function will be the solution function in the region that is the domain for the transformation function. An example in physics will be used to help clarify the technique.

**An example in physics.** For this section, an example from Saff & Snider (2016) pages 421-422 is followed. A knowledge of the point at infinity in the complex plane is required for this example.

Example 1. Find the function $\phi$ that is harmonic in the shaded domain, depicted below in Figure 1, and takes the value 0 on the inner circle and 1 on the outer circle. $\phi$ can be interpreted as the electrostatic potential inside a capacitor formed by two nested parallel cylindrical conductors.

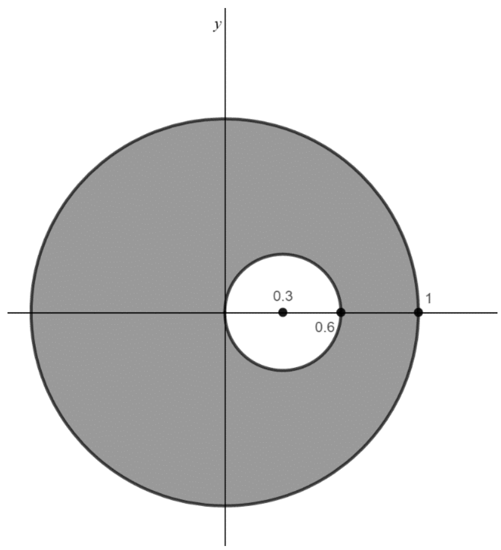


*Figure 1*. Example 1 pre-transformation graph. Graph of the initial region before the conformal mapping technique is used.

To solve this problem, map the given region into a washer so the two circles are concentric. A pair of real points $z = x_1$ and $z = x_2$ that are symmetric, with respect to both circles simultaneously, can be found. For this, a formula, derived in Saff & Snider (2016), relating the two points with respect to some circle is given below where the bars above the numbers are the standard notation for the complex conjugate:

$$x_1 = \frac{R^2}{\overline{x}_2 - \overline{a}} + a \tag{30}$$

where $a$ is the center of the circle and $R$ is the radius. In this case, with respect to the outer circle,

$$x_2 = \frac{1}{x_1}. \tag{31}$$

The symmetry with respect to the inner circle is,

$$x_2 - 0.3 = \frac{(0.3)^2}{x_1 - 0.3}. \tag{32}$$

Solve these equations to find,

$$x_1 = \frac{1}{3}\ , \quad x_2 = 3. \tag{33}$$

Select the Möbius transformation sending $x_1$ to 0 and $x_2$ to $\infty$ via,

$$w = f(z) = \frac{z - 1/3}{3 - z}. \tag{34}$$

The resulting region is a pair of circles where 0 and $\infty$ are symmetric points. It is a fact that the point at infinity is symmetric to the center of a circle, so the resulting region has concentric circles as depicted in Figure 2.

This is a simple washer problem which has template solution $\phi(x, y) = A\log|w| + B$ where $A$ and $B$ are real numbers. The radius of the inner circle is found to be,

$$|w_i| = |f(0)| = \frac{1}{9}. \tag{35}$$

The outer circle has radius,

$$|w_o| = |f(1)| = \frac{1}{3}. \tag{36}$$

transformed under function $f(z)$.

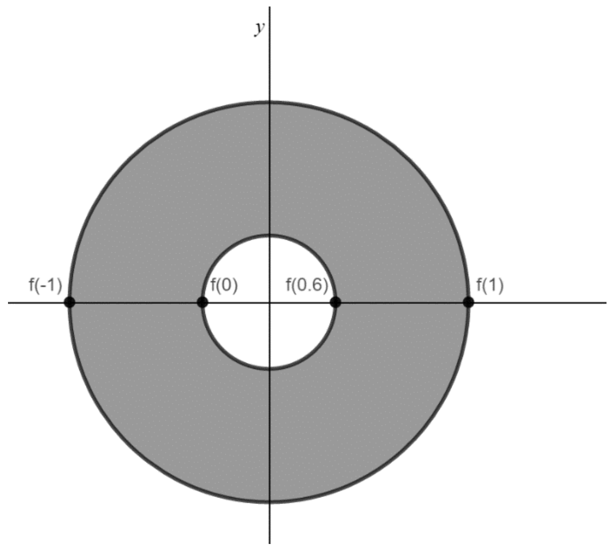


*Figure 2*. Example 1 post-transformation graph. Graph of the region in Figure 1

The solution is found by solving for constants $A$ and $B$ below,

$$A\log|w_i| + B = 1 \qquad A\log|w_o| + B = 0. \tag{37}$$

Solve to obtain the harmonic function,

$$\Psi = \frac{\log|9w|}{\log 3}. \tag{38}$$

Now transform this solution back into the starting region:

$$\phi(z) = \Psi\left(\frac{z - 1/3}{z - 3}\right)$$

$$= \frac{\log\left|\frac{9z-3}{z-3}\right|}{\log 3}$$

$$= \frac{1}{\log 3}\left[\log 3 + \frac{1}{2}\log[(3x-1)^2 + 9y^2] - \frac{1}{2}\log[(x-3)^2 + y^2]\right]. \tag{39}$$

Solving the above problem directly would have been extremely difficult as the solution suggests. By using the technique of conformal mapping, the complicated problem was moved into a simplified region where a solution was already known. The constants for the solution function were found and then the solution was mapped back into the more complicated region to produce the desired solution. The list of examples demonstrating this method are endless and can be found in any complex-analysis text that covers conformal mapping.

## Fundamental Theory of Associative Real Algebras

The focus of the paper will shift now to the study of hyperbolic numbers and the development of a solution technique for the wave equation like conformal mapping. The needed results from Cook (2017) will be stated and then revisited in the next section when it is proven that the hyperbolic numbers form an associative real algebra and thus can utilize the stated theorems from this section. A working knowledge of Linear Algebra and Advanced Calculus is assumed for the remainder of the paper. See Curtis (1984) for an excellent resource in Linear Algebra, and Edwards (1994), and McInerney (2013) for excellent resources in Advanced Calculus.

Definition 12. Let $\mathcal{A}$ be a finite-dimensional real vector space paired with a function $\star : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is called a multiplication. The multiplication map $\star$ satisfies the below properties:

1. Bilinear**:** $(cx + y) \star z = c(x \star z) + y \star z$ and $x \star (cy + z) = c(x \star y) + x \star z$ for all $x, y, z \in \mathcal{A}$ and $c \in \mathbb{R}$.
2. Associative**:** $x \star (y \star z) = (x \star y) \star z$ for all $z, y, z \in \mathcal{A}$.

3. Unital**:** There exists $1 \in \mathcal{A}$ for which $1 \star x = x$ and $x \star 1 = x$.

If $x \star y = y \star x$ for all $x, y \in \mathcal{A}$ then $\mathcal{A}$ is commutative.

The above construction is called an Algebra. Given the structure above, notice that an algebra is a vector space paired with a multiplication.

Definition 13. If $\alpha \in \mathcal{A}$ then $L_\alpha(x) = \alpha \star x$ is a left-multiplication map on $\mathcal{A}$. It is right-$\mathcal{A}$-linear as,

$$L_\alpha(x \star y) = \alpha \star (x \star y) = (\alpha \star x) \star y = L_\alpha(x) \star y. \tag{40}$$

Definition 14. Let $\mathcal{R}_\mathcal{A}$ define the set of all right-$\mathcal{A}$-linear transformations on $\mathcal{A}$. If $T \in \mathcal{R}_\mathcal{A}$ then $T: \mathcal{A} \to \mathcal{A}$ is an $\mathbb{R}$-linear transformation for which $T(x \star y) = T(x) \star y$ for all $x, y \in \mathcal{A}$.

Given an algebra $\mathcal{A}$ with basis $\beta = \{1, v_2, v_3, \dots, v_n\}$ we can define an inner product on $\mathcal{A}$. The details of this inner product can be found on pages 16-17 of Cook (2017). With the norm defined, the definition of differentiability follows.

Definition 15. Let $U \subseteq \mathcal{A}$ be an open set containing $p$. If $f: U \to \mathcal{A}$ is a function, then we say $f$ is $\mathcal{A}$-differentiable at $p$ if there exists a linear function $d_p f \in \mathcal{R}_\mathcal{A}$ such that

$$\lim_{h \to 0} \left( \frac{f(p+h) - f(p) - d_p f(h)}{\left| |h| \right|} \right) = 0. \tag{41}$$

The above definition is the Fréchet limit presented in any advanced calculus course.

Theorem 9. If $f$ is $\mathcal{A}$-differentiable at $p$ then $f$ is $\mathbb{R}$-differentiable at $p$.

Proof: Following Cook's proof on page 18 (2017), if $f$ is $\mathcal{A}$-differentiable at $p$ then $d_p f \in \mathcal{R}_\mathcal{A}$ satisfies the Fréchet limit given in Definition 15. Thus $f$ is $\mathbb{R}$-differentiable. ■

Using the results presented in this section, the solution technique for solving the wave equation via transformation mappings can be developed. The full construction of the technique is given in the next section with references back to Cook's paper for the proofs that are outside the scope of this paper.

## Transformations in the Hyperbolic Plane

A solution technique like conformal mapping is found in the next section of the paper. In studying the properties of the hyperbolic numbers, we discover a series of transformation functions that preserve solutions to the wave equation when mapping from region to region. A catalog of key transformations and solutions will be given to conclude.

### The Hyperbolic Numbers

To begin the discussion, defining the hyperbolic plane is needed. Proving that they satisfy the conditions of an associative real algebra, and therefore the results in the previous section will assist in the development of the wave equation solution technique.

Definition 16. Define the hyperbolic number system as the set $\mathcal{H}$ where,

$$\mathcal{H} = \{a + jb : j^2 = 1 \quad a, b \in R\} \tag{42}$$

Given $z, w \in \mathcal{H}$ where $z = x + jy$ and $w = a + jb$, define addition and subtraction in the following way,

$$z + w = (x + a) + j(y + b) \ , \quad zw = (xa + yb) + j(xb + ya). \tag{43}$$

Having this definition, consider elements $z$ within $\mathcal{H}$ where $z = x + jy$. Define the conjugate of $z$ as $\bar{z} = x - jy$ and the following results are obtained.

Theorem 10. Let $z, w \in \mathcal{H}$ such that $z = x + jy$. Then we have the following properties,

1. $z\bar{z} = x^2 - y^2$,
2. $\overline{zw} = \bar{z}\bar{w}$,
3. $z^{-1} = \frac{\bar{z}}{x^2 - y^2}$.

Proof: For each statement in the theorem, Definition 17 can be used to show them directly. Suppose $\mathrm{z}, \mathrm{w} \in \mathcal{H}$ then notice $\mathrm{z\bar{z}} = (\mathrm{x} + \mathrm{jy})(\mathrm{x} - \mathrm{jy}) = \mathrm{x}^2 - \mathrm{y}^2$ which proves (10.1). Next, notice $\overline{\mathrm{zw}} = \overline{(\mathrm{x} + \mathrm{jy})(\mathrm{e} + \mathrm{jf})} = \mathrm{xe} + \mathrm{yf} - \mathrm{j}(\mathrm{xf} + \mathrm{ye}) = (\mathrm{x} - \mathrm{jy})(\mathrm{e} - \mathrm{jf}) = \overline{\mathrm{z}}\overline{\mathrm{w}}$ which proves (10.2). Finally, observe that $1/\mathrm{z} = \overline{\mathrm{z}}/(z\overline{z}) = \overline{\mathrm{z}}/(\mathrm{x}^2 - \mathrm{y}^2)$ by (10.1) which proves (10.3). ■

With the above definitions, we can show that the hyperbolic numbers form an associative real algebra. Consider $\alpha \in \mathbb{R}$ and $x, y, z \in \mathcal{H}$. Thus, $x = a + bj$, $y = c + dj$, and $z = e + fj$ for $a, b, c, d, e, f \in \mathbb{R}$. The standard multiplication will be the operation denoted by juxtaposition. Notice,

$$
\begin{aligned}
(\alpha x + y)z &= \big(\alpha(a + bj) + (c + dj)\big)(e + fj) \\
&= (\alpha a + \alpha bj + c + dj)(e + fj) \\
&= (\alpha ae + ce + \alpha bf + df) + j(\alpha af + cf + \alpha be + de) \\
&= \alpha\big((ae + bf) + j(af + be)\big) \\
&= \alpha\big((a + bj)(e + fj)\big) + (c + dj)(e + fj) \\
&= \alpha(xz) + yz.
\end{aligned}
$$

A similar argument is used to show $x(cy + z) = \alpha(xy) + xz$. Thus, the hyperbolic numbers are bilinear. Showing the hyperbolic numbers are associative is trivial, and there exists $1 \in \mathcal{H}$ where $1 \star x = x$ and $x \star 1 = x$ for $x \in \mathcal{H}$. Thus, the hyperbolic numbers are unital. Therefore, the hyperbolic numbers form an associative real algebra. It can also be shown that the hyperbolic numbers are commutative.

**Hyperbolic-Differentiable Functions**

By replacing $\mathcal{A}$ with $\mathcal{H}$ in Definition 15, the differentiability of hyperbolic-valued functions is constructed.

Definition 17. Let $U \subseteq \mathcal{H}$ be an open set containing $p$. If $f: U \to \mathcal{H}$ is a function, then we say $f$ is $\mathcal{H}$-differentiable at $p$ if there exists a linear function $d_p f: \mathcal{H} \to \mathcal{H}$ such that

$$\lim_{h \to 0} \left( \frac{f(p+h) - f(p) - d_p f(h)}{\left||h|\right|} \right) = 0. \tag{44}$$

If $f$ is $\mathcal{H}$-differentiable at every point $p \in U$ then we say $f$ is $\mathcal{H}$-differentiable on $U$.

The definition is again a direct result of the definition of the Fréchet Differential from advanced calculus. Following the path laid out in the previous section on associative real algebras, Theorem 9 can be constructed with the hyperbolic numbers in the following way.

Theorem 11. If $f$ is $\mathcal{H}$-differentiable at $p \in U \subseteq \mathcal{H}$ then $f$ is $\mathbb{R}$-differentiable at $p \in U$.

Proof: As the hyperbolic numbers form an algebra, Theorem 9 can be directly applied to prove Theorem 11. ■

In a similar construction to the complex numbers, the Hyperbolic Cauchy-Riemann equations can be constructed as follows.

Theorem 12. If a function $f = u + jv$ is $\mathcal{H}$-differentiable at $p \in U \subseteq \mathcal{H}$ then $u_x = v_y$ and $u_y = v_x$.

Proof: Suppose $f = u + jy$ is $\mathcal{H}$-differentiable at $p$. Then notice $\text{span}\{1, j\} = \mathcal{H}$. This implies that $\beta = \{1, j\}$ is a basis for $\mathcal{H}$; in fact, it is the standard basis $\beta = \{e_1, e_2\}$ where $e_1 = 1$ and $e_2 = j$. Since $f$ is $\mathcal{H}$-differentiable we have the function $d_p f$ by Definition 17. Consider the Jacobian matrix of $d_p f$,

$$[d_p f] = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}. \tag{45}$$

Now we look at the separate basis elements with respect to the Jacobian matrix,

$$(d_p f)(e_1) = (d_p f)(1) = \begin{bmatrix} u_x \\ v_x \end{bmatrix}, \qquad \text{and} \tag{46}$$

$$(d_p f)(e_2) = (d_p f)(j) = \begin{bmatrix} u_y \\ v_y \end{bmatrix}. \tag{47}$$

Here we observe that $j = 1 \cdot j$ and obtain the following,

$$(d_p f)(e_2) = (d_p f)(1 \cdot j) = \left((d_p f)(1)\right) j = (u_x + v_x j) j = v_x + u_x j = \begin{bmatrix} v_x \\ u_x \end{bmatrix} = \begin{bmatrix} u_y \\ v_y \end{bmatrix}. \tag{48}$$

Thus, we find the desired result $u_x = v_y$ and $u_y = v_x$. An identical argument can be made in the context of the complex numbers using the standard basis $\{1, i\}$. ■

The Jacobian matrix is explained in full detail on pages 188-194 of C.H. Edwards' book *Advanced Calculus of Several Variables* (1994).

With this in place, a theorem for the sufficiency of $\mathcal{H}$-differentiability of a function follows naturally.

Theorem 13. If a function $f = u + jv$ is continuously $\mathbb{R}$-differentiable ($u_x, u_y, v_x, v_y$ all continuous), at a point $p \in U \subseteq \mathcal{H}$ and $u_x = v_y$ & $u_y = v_x$ at $p$ then $f$ is $\mathcal{H}$-differentiable at $p$.

Proof: Suppose $f$ is continuously $\mathbb{R}$-differentiable and the hyperbolic Cauchy-Riemann equations satisfied. Then the hyperbolic Cauchy-Riemann equations being satisfied means that $d_p f$ is right-$\mathcal{H}$-linear which implies that $f$ is $\mathcal{H}$-differentiable. ■

One final theorem is required before discussing the wave equation. The composition of two $\mathcal{H}$-differentiable functions is again $\mathcal{H}$-differentiable.

Theorem 14. Suppose $f$ and $g$ are $\mathcal{H}$-differentiable functions satisfying $f: U \to V$ and $g: V \to W$ where $U, V, W \subseteq \mathcal{H}$ then $g \circ f : U \to W$ is $\mathcal{H}$-differentiable with,

$$\frac{d}{dz}(g \circ f) = \frac{dg}{dz}\big(f(z)\big)\frac{df}{dz}. \tag{49}$$

Proof: Notice, $f$, and $g$ are maps on open subsets of $\mathbb{R}^2$ and are composable $\mathbb{R}$-differentiable maps. Thus,

$$d_p(g \circ f) = \big(d_{f(p)}g\big) \circ d_p f. \tag{50}$$

To show $g \circ f$ is $\mathcal{H}$-differentiable we need to demonstrate right-$\mathcal{H}$-linearity:

$$d_p(g \circ f)(vw) = \big(d_{f(p)}g\big)\Big(d_p f(vw)\Big) = \big(d_{f(p)}g\big)\big(d_p f(v)w\big) \tag{51}$$

as $f$ is $\mathcal{H}$-differentiable at $p \in U$. Then,

$$= \Big(\big(d_{f(p)}g\big)\big(d_p f\big)(v)\Big)(w) \tag{52}$$

as $g$ is $\mathcal{H}$-differentiable at $p \in U$. Then,

$$= \Big(d_p(g \circ f)(v)\Big)(w). \tag{53}$$

Further details can be found on pages 22-24 of Cook's paper (2017). ■

**The Wave Equation**

In a previous section complex conformal mapping was developed which naturally led to obtaining the standard Cauchy-Riemann equations. It was also found that every complex differentiable function, or analytic function, $f = u + iv$ has component functions $u$ and $v$ that satisfy the Laplace differential equations $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

Following an almost identical argument, similar results for the hyperbolic numbers are found but with the one-dimensional wave equation.

Theorem 15. If a function $f = u + jv$ is $\mathcal{H}$-differentiable at $p \in U \subseteq \mathcal{H}$ then,

$$u_{xx} - u_{yy} = 0, \text{ and } v_{xx} - v_{yy} = 0. \tag{54}$$

Proof: Suppose a function $f = u + jv$ is $\mathcal{H}$-differentiable at $p \in U \subseteq \mathcal{H}$. Notice by Theorem 12 that $u_x = v_y$ and $u_y = v_x$. In standard partial differential notation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{55}$$

Starting with the left-hand equation notice,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}. \tag{56}$$

But partial derivatives commute so,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x}. \tag{57}$$

Then by Equation 55,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \tag{58}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \tag{59}$$

Changing back to the original notation,

$$u_{xx} - u_{yy} = 0. \tag{60}$$

By a similar argument,

$$v_{xx} - v_{yy} = 0. \tag{61}$$

Thus, the theorem is proved. ■

The component functions $u$ and $v$ of an $\mathcal{H}$-differentiable function $f$ are both solutions to the so-called Hyperbolic Laplace Equations, or more precisely, the one-dimensional wave equation. The one-dimensional wave equation is typically written,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{62}$$

Where $c$ is some fixed constant and $t$ is time. In the context of this paper, $c = 1$ and the variables $t, x$ are replaced with $x, y$. Solving the wave equation with $\mathcal{H}$-differentiable functions is an extremely powerful tool and allows the study of specific regions in the hyperbolic number plane that by construction allow the wave equation to be solved on them. To demonstrate this feature of the hyperbolic numbers, some of the basic functions for the hyperbolic numbers will be defined and a demonstration that their component functions solve the wave equation will be given.

**Solutions to common functions.** This section will cover four common functions that find great use when looking at transformations in the hyperbolic plane. This section will examine the exponential function, the logarithm, the reciprocal function and the square function. Each function will be proven to be hyperbolic-differential and its components verified as solutions to the wave equation.

Theorem 16. Consider a hyperbolic number $z = x + jy$. Then the exponential function can be written,

$$e^z = e^x(\cosh y + j\sinh y). \tag{63}$$

Proof: Consider the Taylor series of $e^{jx}$ where $x \in \mathbb{R}$,

$$e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \cdots. \tag{64}$$

Also, notice the series expansions of $\cosh(x)$ and $\sinh(x)$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \text{and} \tag{65}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots. \tag{66}$$

Given this, we can see,

$$\begin{aligned} e^{jx} &= 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5} + \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + j\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cosh(x) + j\sinh(x). \end{aligned} \tag{67}$$

Consider for $z = x + jy$,

$$e^z = e^{x+jy} = e^x e^{jy} = e^x\big(\cosh(x) + j\sinh(x)\big). \tag{68}$$

Thus, the identity holds true. ∎

Having the above identity provides a way to break the exponential function into its component functions. Before proceeding, we need to ensure that the exponential is indeed a function and does not require special treatment as in the complex case with branches. First, notice that the exponential is into the hyperbolic plane.

Secondly, suppose,

$$e^{z_1} = e^{x_1}\cosh(y_1) + je^{x_1}\sinh(y_1) = e^{x_2}\cosh(y_2) + je^{x_2}\sinh(y_2) = e^{z_2}. \quad (69)$$

From this notice,

$$e^{x_1}\cosh(y_1) = e^{x_2}\cosh(y_2), \text{and} \quad (70)$$

$$e^{x_1}\sinh(y_1) = e^{x_2}\sinh(y_2). \quad (71)$$

Divide these two equations to obtain,

$$\tanh(y_1) = \tanh(y_2). \quad (72)$$

As $\tanh(x)$ is a non-periodic function, $y_1 = y_2$ and thus $e^z$ is a 1-1 function.

Now that $e^z$ is established as an injective function, notice that for $z = x + jy$,

$$e^z = e^{x+jy} = e^x\cosh(y) + je^x\sinh(y) = u + jv \quad (73)$$

where $x, y \in \mathbb{R}$. We have that $u = e^x\cosh(y)$ and $v = e^x\sinh(y)$. Next, consider the partial derivatives of $u$ and $v$,

$$u_x = e^x\cosh(y), and\ u_y = e^x\sinh(y), \quad (74)$$

$$v_x = e^x\sinh(y), and\ v_y = e^x\cosh(y). \quad (75)$$

Notice that $u_x, u_y, v_x$, and $v_y$ are all continuous, implying that $e^z$ is $\mathbb{R}$-differentiable. Also, notice that $u_x = v_y$ and $u_y = v_x$ so Theorem 13 applies, and we conclude that $e^z$ is $\mathcal{H}$-differentiable. Because of this, by Theorem 15, $u = e^x\cosh(y)$ and $v = e^x\sinh(y)$ are both solutions to the wave equation.

The Logarithm Function is constructed from the results found on the exponential function. Let $w = e^z$ where $z \in \mathcal{H}$ then define $\log(w) = z$. It was shown that $e^z$ is a 1-1 function, so the $\log(w)$ function should act naturally.

Suppose $z = x + jy$ and $w = u + jv$ where $z, w \in \mathcal{H}$ then,

$$w = e^z = e^x \cosh(y) + je^x \sinh(y) = u + jv \tag{76}$$

$$\Rightarrow \frac{e^x \sinh(y)}{e^x \cosh(y)} = \frac{v}{u} = \tanh(y) \tag{77}$$

As $e^x \cosh(y) > 0$. Thus,

$$y = \tanh^{-1}\left(\frac{v}{u}\right). \tag{78}$$

Now for $x$ observe that,

$$u^2 - v^2 = e^{2x}\left(\cosh^2(y) - \sinh^2(y)\right) = e^{2x} \tag{79}$$

since $\cosh^2(y) - \sinh^2(y) = 1$ (Briggs et al., 2013). This provides,

$$x = \ln\sqrt{u^2 - v^2}. \tag{80}$$

Having found $x$ and $y$, put them together and we find that,

$$\log(w) = \ln\sqrt{u^2 - v^2} + j\tanh^{-1}\left(\frac{v}{u}\right). \tag{81}$$

Garret Sobczyk (1995) points out in his research that the $\tanh^{-1}\left(\frac{v}{u}\right)$ portion of this formula only holds in quadrants one and three of the hyperbolic plane due to the existence of zero divisors.

Let $\log(w) = x + jy$, and notice that $x$ and $y$ are real-valued functions. After some calculation, it can be shown that $\log(w)$ satisfies the Hyperbolic Cauchy-Riemann equations and by Theorem 13 we conclude that it is $\mathcal{H}$-differentiable. Therefore, by

Theorem 15, the component functions of $\log(w)$ are both solutions to the wave equation. For the examples in the final section of this paper, the arctanh component of the log function will be the focus as it provides an excellent template solution for transformations to map regions into.

The reciprocal function, $f(z) = 1/z$ can be shown to be $\mathcal{H}$-differentiable and has component functions,

$$\frac{1}{z} = \frac{x}{x^2 - y^2} - j\frac{y}{x^2 - y^2}. \tag{82}$$

The square function, $f(z) = z^2$ can also be shown to be $\mathcal{H}$-differentiable and has component functions,

$$z^2 = x^2 + y^2 + j(2xy). \tag{83}$$

The reader can verify that each of the component functions listed are solutions to the wave equation by Theorem 15.

**Basic Hyperbolic Transformations**

Here we deviate from the standard construction of conformal mapping that was shown in the complex numbers and shift the focus to solving the wave equation. With the established theorems, the wave equation can be solved in one region, mapped into another region via an $\mathcal{H}$-differentiable function $f$ where a solution to the wave equation is known, and then mapped back to the starting region. This is the same technique as conformal mapping but in the context of the hyperbolic numbers.

In a similar fashion to complex conformal mapping there exist basic building block transformations: translation, rotation, and inversion. The following examples have an $\mathcal{H}$-differentiable function $f$ evaluated at a point $z = x + jy$, an element of the hyperbolic numbers.

The translation mapping is written,

$$w = f(z) = z + c \tag{84}$$

where $c$ is a fixed hyperbolic number. As in the complex numbers, this transformation simply shifts $z$ by a factor of $c$.

For rotation, a differing approach from the complex numbers is needed. Whereas any complex number $z$ can be written,

$$z = |z|e^{i\theta}, \tag{85}$$

any hyperbolic number can be written as one of the following when it satisfies the various conditions presented below. Let $h = \sqrt{|x^2 - y^2|}$.

1. For the region bounded by $x \geq 0$ and $-x \leq y \leq x$ called segment I,

$$z = he^{j\phi} \tag{86}$$

where $\phi = \tanh^{-1}\left(\frac{y}{x}\right)$.

2. For the region bounded by $y \geq 0$ and $y \leq x \leq -y$ called segment II,

$$z = jhe^{j\phi} \tag{87}$$

where $\phi = \tanh^{-1}\left(\frac{x}{y}\right)$.

3. For the region bounded by $x \leq 0$ and $-x \geq y \geq x$ called segment III,

$$z = -he^{j\phi} \tag{88}$$

where $\tanh^{-1}\left(\frac{y}{x}\right)$.

4. For the region bounded by $y \leq 0$ and $y \leq x \leq -y$ called segment IV,

$$z = -jhe^{j\phi} \tag{89}$$

where $\tanh^{-1}\left(\frac{x}{y}\right)$.

The hyperbolic polar coordinates are defined in detail in the next section.

With the above, there are two rotation functions used for moving between the bounded regions above. The function used for rotation by 90 degrees counterclockwise is,

$$w = f(z) = jz. \tag{90}$$

The function used for rotation by 180 degrees counterclockwise is,

$$w = f(z) = -z. \tag{91}$$

From these two functions, we derive a rotation by 270 degrees counterclockwise with,

$$w = f(z) = -jz. \tag{92}$$

The final transformation is the inversion transformation written,

$$w = f(z) = \frac{1}{z}. \tag{93}$$

Notice, to transform any function with $f(z) = 1/z$ where $z$ is a complex number, just solve for the real values $x$ and $y$. Consider,

$$f(z) = \frac{1}{z} = \frac{1}{x + jy} = \frac{1}{x + jy} * \frac{x - jy}{x - jy} = \frac{x}{x^2 - y^2} - j\frac{y}{x^2 - y^2}. \tag{94}$$

The inverse mapping takes each $x$ value to $x/(x^2 - y^2)$ and each $y$ value to $-y/(x^2 - y^2)$. Applying this to the unit hyperbola equation we find,

$$x^2 - y^2 = 1 \rightarrow \left(\frac{x}{x^2 - y^2}\right)^2 - \left(\frac{-y}{x^2 - y^2}\right)^2 = 1 \rightarrow \frac{x^2 - y^2}{(x^2 - y^2)^2} = 1 \rightarrow x^2 - y^2 = 1. \tag{95}$$

This is the action seen with complex inversion sending regions to circles (Saff & Snider, 2016). To assist in understanding the rotation mappings in the hyperbolic numbers, a polar coordinate representation of any hyperbolic number is defined. This feature of the hyperbolic numbers has the potential to reduce the complexity of problems.

**Hyperbolic Polar Coordinates**

The polar representation of the complex numbers is an extremely useful convention that makes solving certain equations significantly easier. For the hyperbolic numbers, a similar polar coordinate system that will assist in the solving of complicated equations can be defined. To achieve this, revisiting the structure of the hyperbolic numbers is needed.

A nice feature of the complex numbers is that the modulus of any complex number $z$ corresponds to the magnitude of that numbers vector in the complex plane which can then be used to represent numbers in polar coordinates. Unfortunately, in the hyperbolic plane, this luxury is not afforded, so an alternative construction must take place. As is done with complex numbers, think of hyperbolic numbers as vectors in the hyperbolic plane, each with a magnitude and direction. Represent this magnitude and direction in the hyperbolic plane by splitting it into four segments separated by the lines of $y = x$ and $y = -x$. The four segments will be referred to as segment I, II, II and IV. The hyperbolic plane with this partitioning is shown in Figure 3 with the appropriate labels. Due to the presence of zero divisors in the hyperbolic plane, polar coordinates for each section of the hyperbolic plane must be defined individually.

Referring again to hyperbolic numbers as vectors on the hyperbolic plane, define $h = \sqrt{|x^2 - y^2|}$ to be the magnitude of the vectors, and $\phi$ to be their angle of rotation defined on all of $\mathbb{R}$. Using these two definitions as building blocks, construct a polar coordinate system for each of the four sections that the hyperbolic plane is split into.

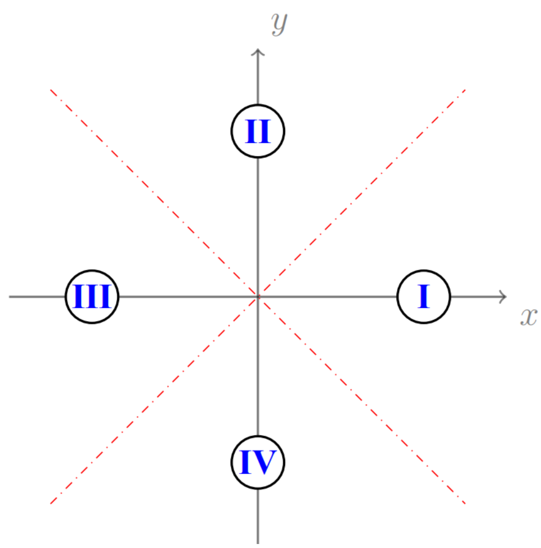


*Figure 3*. Sections of the hyperbolic plane. This figure depicts the four sections separated by zero divisors in the hyperbolic plane.

For the first section (I), define the coordinates to be,

$$x = h\cosh(\phi), \text{ and } y = h\sinh(\phi) \tag{96}$$

which yields $x^2 - y^2 = h^2$ using identities on the hyperbolic functions sinh and cosh (BeDell, 2017c). Solve for $\phi$ in this case to obtain the equation,

$$\phi = \tanh^{-1}\left(\frac{y}{x}\right). \tag{97}$$

This says that $\phi = 0$ whenever $y = 0$ which graphically means that $\phi = 0$ along the $x$-axis of the hyperbolic plane in section (I). Given any hyperbolic number $z = x + jy$ in section (I), convert into polar coordinates to obtain,

$$z = x + jy = h\cosh(\phi) + jh\sinh(\phi) = h\big(\cosh(\phi) + j\sinh(\phi)\big) = he^{j\phi}. \qquad (98)$$

Thus, in section (I), the polar representation of any number is $he^{j\phi}$.

To construct the polar coordinates for the other three sections, take the coordinates in section (I) and transform them to fit the other sections. This is done using the transformation functions $f(z) = jz$, $f(z) = -z$, and $f(z) = -jz$. Knowing that $f(z) = jz$ acts as a rotation by 90 degrees, $f(z) = -z$ acts as rotation by 180 degrees, and $f(z) = -jz$ acts as a rotation by 270 degrees, construct polar coordinates for all of the hyperbolic plane with the exception of the dividing lines $y = x$ and $y = -x$.

For section (II), the polar coordinate mapping must be,

$$z = jhe^{jh} = jh\big(\cosh(\phi) + j\sinh(\phi)\big) = h\big(\sinh(\phi) + j\cosh(\phi)\big) \qquad (99)$$

because this is the original polar map transformed with the $f(z) = jz$ transformation. With this new coordinate map notice,

$$x = h\sinh(\phi), \qquad y = h\cosh(\phi), \qquad \text{and } \phi = \tanh^{-1}\left(\frac{x}{y}\right). \qquad (100)$$

This means that $\phi = 0$ along the $y$-axis in section (II).

For section (III), the polar coordinate mapping must be,

$$z = -he^{j\phi} = -h\big(\cosh(\phi) + j\sinh(\phi)\big). \qquad (101)$$

This is the original polar map under the $f(z) = -z$ transformation. Notice,

$$x = -h\cosh(\phi), \qquad y = -h\sinh(\phi), \text{and } \phi = \tanh^{-1}\left(\frac{y}{x}\right). \qquad (102)$$

Thus $\phi = 0$ along the $x$-axis in section (III).

For section (IV), the polar coordinate mapping must be,

$$z = -jhe^{j\phi} = -jh\big(\cosh(\phi) + j\sinh(\phi)\big) = -h\big(\sinh(\phi) + j\cosh(\phi)\big) \qquad (103)$$

which implies that,

$$x = -h\sinh(\phi), \qquad y = -h\cosh(\phi), \text{and } \phi = \tanh^{-1}\left(\frac{x}{y}\right). \qquad (104)$$

Thus $\phi = 0$ along the $y$-axis in section (IV). Figure 4 summarizes the above results just derived.

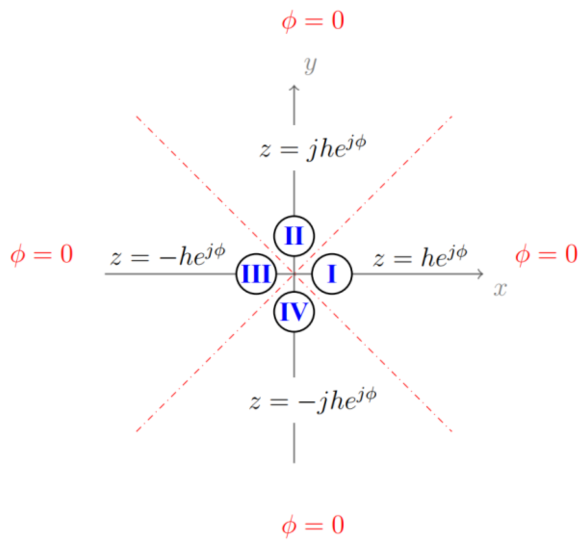


*Figure 4*. Hyperbolic polar coordinates Example 2. This figure displays the polar coordinate maps for each section in the hyperbolic plane.

The polar coordinate system is demonstrated in the following example.

Example 2. Consider hyperbolic numbers $2 + j$, $-2 + 3j$, $1 - 2j$, and $-2 - j$. Plot each number in the hyperbolic plane and then convert each number to polar coordinates and plot them again. First, plot each of the points by writing them as ordered pairs, i.e. (2, 1), (-2, 3), (1, -2), and (-2, -1). This provides the plot shown in Figure 5.

Now write each point in its coordinate system based on which section it resides in. The point $2 + j$ goes to $\sqrt{3}e^{j(0.55)}$, $-2 + 3j$ goes to $\sqrt{5}je^{j(-0.80)}$, $1 - 2j$ goes to $-\sqrt{3}je^{j(-0.55)}$, and $-2 - j$ goes to $-\sqrt{3}e^{j(0.55)}$. Plotting the polar coordinates provides Figure 6.

Comparing both figures, we see that the points are mapped the same.

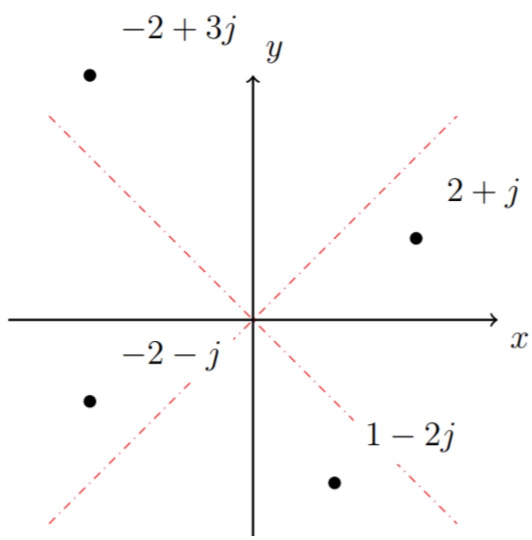


*Figure 5*. Example 2 plot. This figure shows the graphical location of each of the points in Example 2.

Pairing each of the four polar coordinates in each of the sections together with the rotation transformations provides full control over the placement of points in the hyperbolic plane. Now having all the necessary tools, we demonstrate the solution technique for solving the wave equation in various regions.

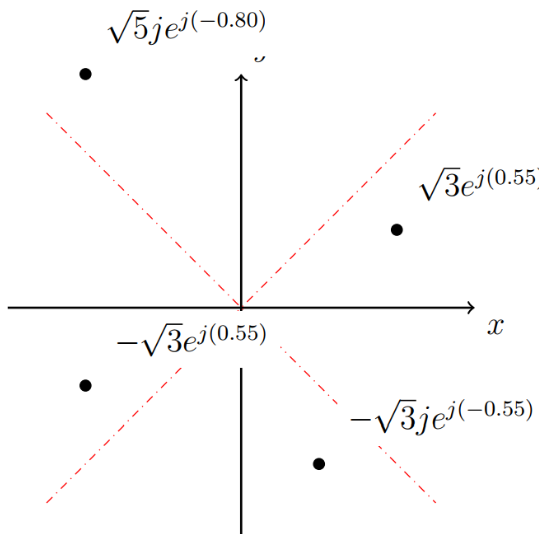


*Figure 6.* Hyperbolic polar coordinates Example 2 post-conversion. This figure shows the values from Figure 5 after they have been transformed via the hyperbolic polar coordinates.

## The Wave Equation Solution Technique

### Basic Example Solutions

Here, the solutions to one-dimensional wave equation boundary value problems are given. The applications of such a problem to physics are not discussed here, but Motter & Rosa (1998) discuss potential physics implications in their paper on hyperbolic calculus. Following an example to demonstrate the solution method, a list of transformation regions will be given to assist in the future study of this solution technique.

Example 3. Consider the shaded region in Figure 7 defined by the inequality $0 \geq y \geq x$ for $x, y \in \mathbb{R}$. Viewing Figure 7 as a vector representation of the hyperbolic number plane, each point $(x, y)$ in the shaded region can be written as a hyperbolic number $z = x + jy$.

We will solve the boundary value problem in which the top curve has value $\phi = 10$ and the bottom curve has value $\phi = 0$. We have shown that $\log(w)$ is $\mathcal{H}$-differentiable and thus its component functions are solutions to the wave equation. We can transform the region into the region given by level curves of $\tanh^{-1}(y/x)$ via the transformation function $f(z) = 1/z$. Transforming the region, we obtain Figure 8 which is defined by $0 \leq y \leq x$.

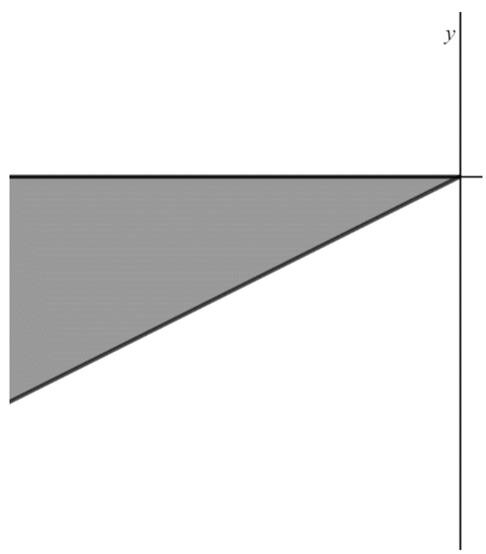


*Figure 7.* Example 3 pre-transformation. This figure shows the region that Example 3 seeks to find a solution to the wave equation on.

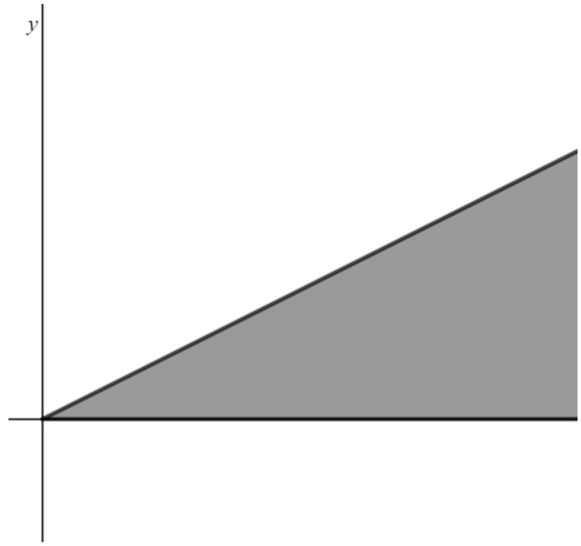


*Figure 8*. Example 3 post-transformation. This figure shows the region in Figure 7 transformed under $f(z)$.

Since $\tanh^{-1}(y/x)$ is a component function of $\log(w)$, it is a solution to the wave equation. Thus, we can use it as a template solution, plug in the boundary conditions and solve the following system of equations.

$$A \tanh^{-1}(y/x) + B = 10 \quad A \tanh^{-1}(y/x) + B = 0. \tag{105}$$

Solve for the value of $\tanh^{-1}(y/x)$ in both above equations. Note the values will not be equal because each equation is describing a different boundary. The leftmost equation is describing the bottom boundary ($\phi = 10$) which is described by the equation $y = 0$. The rightmost equation is describing the top boundary ($\phi = 0$) which is described by the equation $y = x/2$. To find the value of $\tanh^{-1}(y/x)$ in the leftmost equation, we use the following process.

$$\tanh^{-1}(y/x) = c_1 \rightarrow y = \tanh(c_1)x = \frac{x}{2} \Rightarrow c_1 = 0.55. \tag{106}$$

For the rightmost equation,

$$\tanh^{-1}(y/x) = c_2 \to y = \tanh(c_2)x = 0 \Rightarrow c_2 = 0. \tag{107}$$

Now the system to solve for the solution is,

$$A(0.55) + B = 10 \ , \quad B = 0, \tag{108}$$

which provides solution,

$$\Phi = 18.18 \tanh^{-1}\left(\frac{y}{x}\right). \tag{109}$$

Now simply transform this solution back into the starting region,

$$\phi = \Phi\big(f(z)\big) = \Phi\left(\frac{1}{z}\right) = -18.18 \tanh^{-1}\left(\frac{y}{x}\right). \tag{110}$$

Example 4. Consider the transformation function $f(z) = jz$. Which region would be mapped into the region solved by $\Phi = A\tanh^{-1}(y/x) + B$? To find the solution $\phi$, take the composition of the transform function and $f$ and $\Phi$.

$$\phi = \Phi\big(f(z)\big) = \Phi(jz) = \Phi(y + jx) = A\tanh^{-1}\left(\frac{x}{y}\right) + B. \tag{111}$$

Thus transforming $\phi$ under $f$ is $\Phi$. Figure 9 transformed under $f(z)$ is Figure 10.

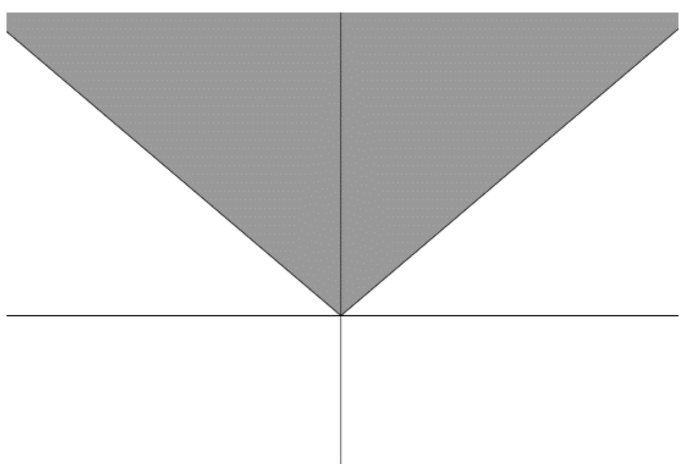

*Figure 9.* Example 4 region pre-transformation. This figure depicts the region that will map into Figure 10 under $f(z)$.

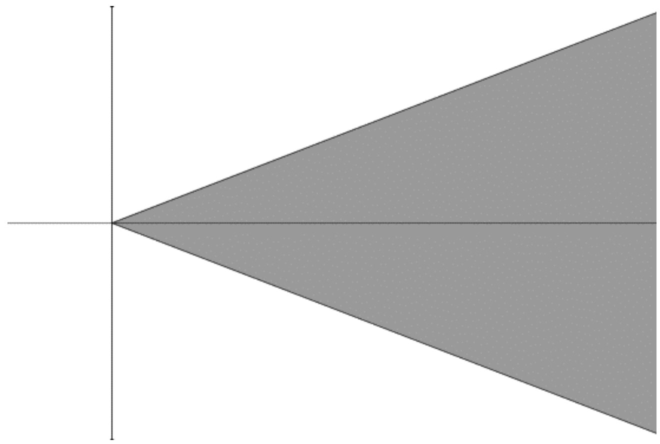

*Figure 10.* Example 4 region post-transformation. This figure depicts the region created by mapping Figure 9 under $f(z)$.

Thus, $\phi$ solves the wave equation on Figure 9, and $\Phi$ solves the wave equation on Figure 10.

Example 5. Consider the transformation function $f(z) = 1/z$. Which region would be mapped into the region solved by $\Phi = A\tanh^{-1}(y/x) + B$? To find the solution $\phi$, take the composition of the transform function and $f$ and $\Phi$.

$$\phi = \Phi\big(f(z)\big) = \Phi\left(\frac{1}{z}\right) = \Phi\left(\frac{x}{x^2 - y^2} - j\frac{y}{x^2 - y^2}\right) = -A\tanh^{-1}\left(\frac{y}{x}\right) + B. \qquad (112)$$

Thus transforming $\phi$ under $f$ is $\Phi$. Figure 11 transformed under $f(z)$ is Figure 12. Therefore $\phi$ solves the wave equation on Figure 11, and $\Phi$ solves the wave equation on Figure 12.

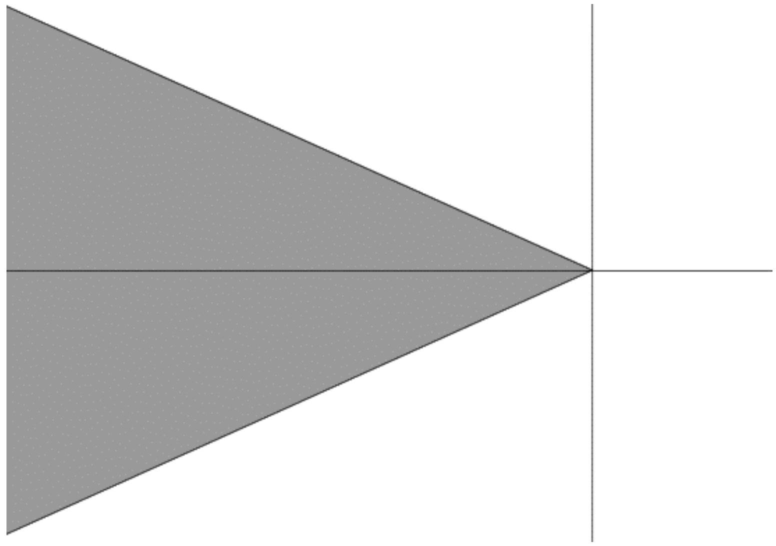

*Figure 11*. Example 5 region pre-transformation. This figure depicts the region that will map into Figure 12 under $f(z)$.

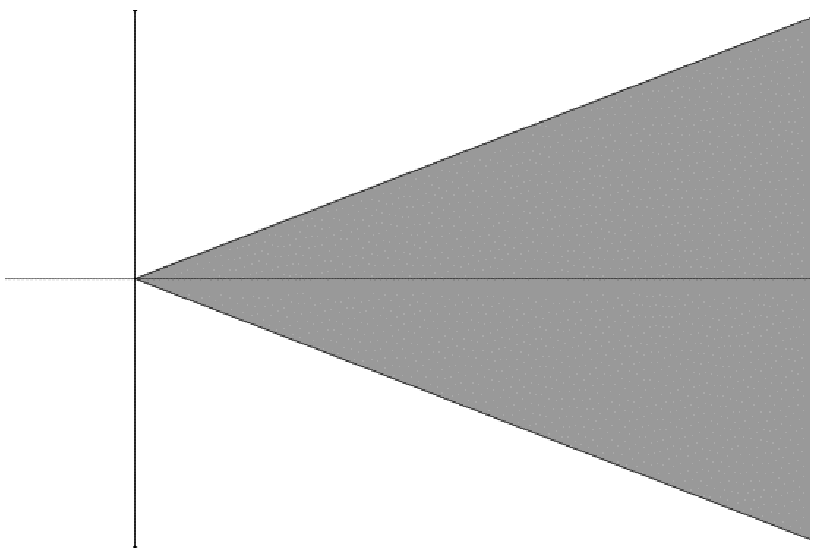

*Figure 12*. Example 5 region post-transformation. This figure depicts the region created by mapping Figure 11 under $f(z)$.

Example 6. Consider the transformation function $f(z) = z^2$. Which region would be mapped into the region solved by $\Phi = A \tanh^{-1}(y/x) + B$? To find the solution $\phi$, take the composition of the transform function and $f$ and $\Phi$.

$$\phi = \Phi\big(f(z)\big) = \Phi(z^2) = \Phi\big(x^2 + y^2 + j(2xy)\big) = A \tanh^{-1}\left(\frac{2xy}{x^2 + y^2}\right) + B. \quad (113)$$

Thus transforming $\phi$ under $f$ is $\Phi$. To find the region where $\phi$ is a solution, we need only find the inverse of $f$ and transform the post transformation region under it. The inverse function $f^{-1}(z)$ is seen to be $\sqrt{z}$. Following a calculation in BeDell & Cook (2017), a formula for the square root of a hyperbolic number $z = x + jy$ is shown to be,

$$(x + jy)^{1/2} = \frac{1}{\sqrt{2}}\sqrt{x + \sqrt{x^2 - y^2}} + \frac{j}{\sqrt{2}}\sqrt{x - \sqrt{x^2 - y^2}}. \quad (114)$$

Using this formula, the various points in the post-transformation region shown in Figure 12 can be transformed. First, notice that the top and bottom boundaries of Figure 12 are described by the set of points $\gamma_+ = \{k(1 + j) : k \in R\}$ and $\gamma_- = \{k(1 - j) : k \in R\}$ respectively. Under the transformation $f^{-1}(z) = \sqrt{z}$ we find that $\gamma_+ = \{\sqrt{k/2}(1 + j)\} = \gamma_-$. Thus, both boundaries are mapped to the same position as the upper boundary with the length compressed by a factor of $\sqrt{k/2}$. To study the points between the two boundaries the polar representation is useful. Recall that in the region being considered, each point $u + jv$ can be written $he^{j\phi}$ where $h = \sqrt{u^2 - v^2}$. Each point under $f^{-1}$ is then calculated to be $\sqrt{he^{j\phi}} = \sqrt{h/2}\big(\sqrt{\cosh\phi + 1} + j\sqrt{\cosh\phi - 1}\big)$. Points below the upper boundary $\gamma_+$ and above the $x$ axis are represented by the angle $\phi_1 = \cosh^{-1}A$ for some value $A$ in the domain of cosh. Similarly, angle $\phi_2 = -\cosh^{-1}A$ represented points below the $x$ axis and above $\gamma_-$. As cosh is an even function, we have that $\phi_1 = \phi_2$. Plugging these values into the arbitrary point given above, we find that $f^{-1}$ takes points in Figure 12 and maps them in such a way as to double cover the region between $\gamma_+$ and the $x$ axis. Figure 13 displays this result.

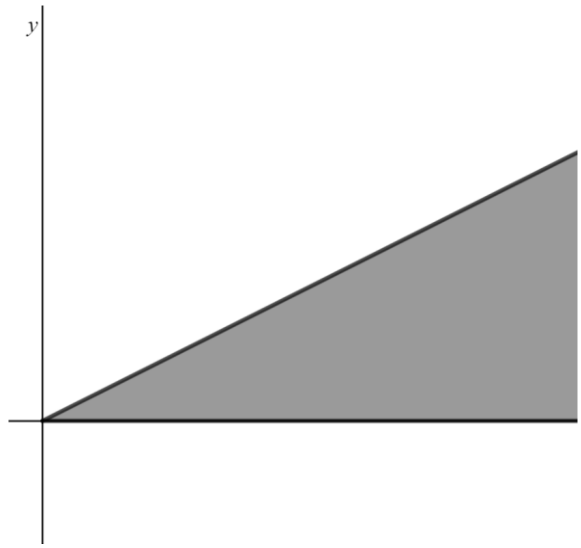


*Figure 13.* Example 6 pre-transformation region. Under the $f = z^2$ mapping, this region would map into the region depicted in Figure 12.

Example 7. Consider the transformation function $f(z) = (j - z)/(z + j)$. Which region would be mapped into the region solved by $\Phi = A\tanh^{-1}(y/x) + B$? To find the solution $\phi$, take the composition of the transform function and $f$ and $\Phi$.

$$\begin{aligned}
\phi &= \Phi\big(f(z)\big) \\
&= \Phi\left(\frac{j - z}{z + j}\right) \\
&= \Phi\left(\frac{x^2 + y^2 - 1}{x^2 - (y + 1)^2} + j\frac{2x}{x^2 - (y + 1)^2}\right)
\end{aligned}$$

$$= A\tanh^{-1}\left(\frac{2x}{x^2+y^2-1}\right) + B. \tag{115}$$

Thus transforming $\phi$ under $f$ is $\Phi$. After a short calculation, the inverse of $f$ is found to be $f(z) = j(1-z)/(1+z)$. Similar calculations can be made as in Example 6 to find the region that the inverse function maps the post-transformation region into.

**Physics Applications**

The examples provided in the previous section pose interesting questions regarding the physical interpretation of the regions created with the hyperbolic mappings. What does solving the wave equation bounded by a region mean in physics? The answer to this question is beyond the scope of this paper, but hopefully, the structures developed in this paper will assist in furthering that understanding.

Rather than solve the wave equation subject to the boundaries presented above, consider solving the wave equation subject to two points with fixed values as the boundary conditions. The solutions will resemble specific waves rather than regions. Specifically, consider solving the wave equation with unit speed,

$$u_{tt} = u_{xx} \tag{116}$$

subject to the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$. The standard template solution to this is found through Fourier Analysis to be,

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nt) + b_n \sin(nx)\sin(nt). \tag{117}$$

For further study into Fourier Analysis, see the works by Gustafson (1980), Haberman (1998), Logan (2015), and Weinberger (1995).

In general, the standard solution can fit any physically reasonable boundary condition and to solve, the coefficients $a_n$ and $b_n$ must be found. A similar template solution can be found for the hyperbolic numbers. This construction paired with the mapping technique described in previous sections allows for interesting and exotic solutions to the wave equation to be found with minimal effort. Consider the function,

$$f_n(z) = \sin(nz) \tag{118}$$

with hyperbolic number $z = x + jt$ and $n$ an integer. To utilize $f$, it must be broken into its component functions. Utilizing results on the sin function from BeDell (2017c) we find,

$$\begin{aligned} f_n(z) &= \sin\big(n(x + jt)\big) \\ &= \sin(nx + njt) \\ &= \sin(nx)\cos(njt) + \sin(njt)\cos(nx) \\ &= \sin(nx)\cos(nt) + j\sin(nt)\cos(nx). \end{aligned} \tag{119}$$

It can be shown that $f$ is $\mathcal{H}$-differentiable and thus its component functions are solutions to the wave equation with unit speed. Notice the real component of $f$ is the same term associated with the $a_n$ coefficient in the standard Fourier template solution shown above. Similarly, consider the function,

$$\begin{aligned} f_n(z) &= \cos(nz) \\ &= \cos\big(n(x + jt)\big) \\ &= \cos(nx + njt) \\ &= \cos(nx)\cos(njt) - \sin(nx)\sin(njt) \\ &= \cos(nx)\cos(nt) - j\sin(nx)\sin(nt). \end{aligned} \tag{120}$$

Notice here that the $j$ component of $f$ is the same term associated with the $b_n$ coefficient in the standard Fourier template solution. Considering both functions, $\sin(nz)$ and $\cos(nz)$ together, a hyperbolic Fourier template is proposed to be,

$$u(x,t) = \sum_{n=1}^{\infty} j\, A_n \sin(nz) + B_n \cos(nz). \tag{121}$$

Using the hyperbolic Euler's equation, this can be simplified further to,

$$u(x,t) = \sum_{n=1}^{\infty} j\, A_n \sin(nz) + B_n \cos(nz) = \sum_{n=1}^{\infty} C_n e^{jnz}. \tag{122}$$

Consider the initial problem, solving the wave equation of unit speed with the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$. By observation, the solution can be seen to be,

$$u(x,t) = \sin(x)\cos(t). \tag{123}$$

Looking more closely, this solution is the real component of the function $f_n(z) = \sin(x + jt) = \sin(z)$. Using this as the region to be mapped into, various transform functions can be analyzed to discover new and interesting solutions to the wave equation.

Recall that the solution $\phi$ can be found via the composition of $\Phi$ and $f$, the transform function. What follows is a series of examples deriving new solutions to the wave equation. What these solutions represent in physics is unknown, but the strong connection to Fourier analysis suggests the connection is significant. Let $\Phi = \sin(w)$ for hyperbolic number $w = u + jv$. Breaking $\Phi$ into its component functions reveals,

$$\Phi = \sin(u)\cos(v) + j\sin(v)\cos(u). \tag{124}$$

Example 8. Consider the transformation function $f(z) = jz$. To find $\phi$ take,

$$\begin{aligned}\phi(z) &= \Phi\big(f(z)\big)\\ &= \Phi(jz)\\ &= \Phi(t+jx)\\ &= \sin(t)\cos(x) + j\sin(x)\cos(t).\end{aligned} \tag{125}$$

The equation $\phi$ is a solution to the wave equation subject to the boundary conditions transformed under $f(z)$.

Example 9. Consider the transformation function $f(z) = z^2$. To find $\phi$ take,

$$\begin{aligned}\phi(z) &= \Phi\big(f(z)\big)\\ &= \Phi(z^2)\\ &= \Phi(x^2+t^2+2jxt)\\ &= \sin(x^2+t^2)\cos(2xt) + j\sin(2xt)\cos(x^2+t^2).\end{aligned} \tag{126}$$

The equation $\phi$ is a solution to the wave equation subject to the boundary conditions transformed under $f(z)$.

Example 10. Consider the transformation function $f(z) = (j-1)/(1+j)$. To find $\phi$ take,

$$\begin{aligned}\phi(z) &= \Phi\big(f(z)\big)\\ &= \Phi\left(\frac{j-1}{1+j}\right)\\ &= \Phi\left(\frac{x^2+y^2-1}{x^2-(y+1)^2} + j\,\frac{2x}{x^2-(y+1)^2}\right)\end{aligned}$$

$$= \sin\left(\frac{x^2+y^2-1}{x^2-(y+1)^2}\right)\cos\left(\frac{2x}{x^2-(y+1)^2}\right)$$

$$+ j\sin\left(\frac{2x}{x^2-(y+1)^2}\right)\cos\left(\frac{x^2+y^2-1}{x^2-(y+1)^2}\right). \tag{127}$$

The equation $\phi$ is a solution to the wave equation subject to the boundary conditions transformed under $f(z)$.

## Future Research

There are many open questions regarding the hyperbolic number plane, but this paper has hopefully exemplified that these questions are worthy of being answered. An important next step would be to discover a hyperbolic Möbius transformation which would allow more complicated examples to be solved and find more meaningful transformations useful in physics. Another useful area to continue studying would be the physical interpretation of one-dimensional wave equation boundary value problems. Utilizing the Hyperbolic Fourier construction allows for the solving of highly exotic wave equation problems. Interpreting what these exotic solutions mean in physics would be a worthy pursuit. The final section of this paper is meant to be used as a guide for the future study of solutions to the wave equation. The examples presented here are not an exhaustive list but serve as a good foundation for more complicated solutions.

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