William Burnside

Theory of Groups of Finite Order and the Burnside Problem

Maleah Mortorff

A Senior Thesis submitted in partial fulfillment
of the requirements for graduation
in the Honors Program
Liberty University
Spring 2016

Acceptance of Senior Honors Thesis

This Senior Honors Thesis is accepted in partial
fulfillment of the requirements for graduation from the
Honors Program of Liberty University.

_____
Timothy Sprano, Ph.D.
Thesis Chair

_____
Evangelos Skoumbourdis, Ph.D.
Committee Member

_____
Harvey Hartman, Th.D.
Committee Member

_____
Brenda Ayres, Ph.D.
Honors Director

_____
Date

Abstract

As one of the most influential founders of modern group theory, William Burnside and his work generated initial interest in the field of group theory. His book *Theory of Groups of Finite Order* was regarded for several decades as the standard measure for group research.  Namely, the General Burnside Problem examines a finitely generated periodic group, questioning whether or not that group must be necessarily finite.  Breakdowns in this general problem led to a definitive negative answer by Evgeny Golod and Tgor Shararevich in 1964, but paved the way for research into specific cases such as the prime exponent.  This thesis will consider the background of Burnside's mathematical expertise, as well as the general, bounded, and restricted cases of Burnside's problem, concluding with a brief overview of his theorem and lemma.

William Burnside:  Theory of Groups of Finite Order and the Burnside Problem

Influential as a founder of modern group theory, William Burnside generated the initial interest that brought group research to the forefront of mathematics. Publishing over one hundred and sixty papers, three books, and serving on countless committees and public offices, he was both a patient teacher and a ruthless researcher (Burnside, Neumann, & Tompson, p. 15). Topics of interest to him included but were not limited to: elliptic functions, hydrodynamics, electromagnetic theory, differential geometry, projective geometry, statistical mechanics, general theory of functions and modular functions, group theory, and probability and statistics (p. 15).

Burnside was characterized by an elegance and conciseness uncommon amongst his peers, and was known to be severe in his high standards yet fruitful in suggestion (Adian, p. 28). While not particularly fond of working with others in terms of group collaboration, on a few separate occasions he broke this trend for the love of his research (Adelmann & Gerbracht, p. 34). Although not always beneficial to his mathematical pursuits, the remains of the correspondence between Burnside and a few elect contemporaries give current readers a look into who he was both personally and academically. His work in finite group theory ignited a generation of interest in the field, and his work *Theory of Groups of Finite Order* was regarded for several decades as the foundational starting point for any work falling under the category groups. Namely, the General Burnside Problem examines a finitely generated, periodic group, and asks whether or not that group must necessarily be finite. While the general problem broke down into a negative answer, later many specific cases such as the prime exponent and others were proven to be true. Relevant topics to be considered in the study of Burnside

include the General Burnside Problem, the Bounded Burnside Problem, and the restricted

case of Burnside's Problem, Burnside's theorem, Burnside's lemma, Burnside's ring, and

other considerable influences he made on the mathematical community in general.

### Burnside's Background

On July 2, 1852, William Burnside was born in London to William Burnside and

Emma (Knight) Burnside (Forsyth, p. 64). At the age of six the younger William and his

family unfortunately suffered the loss of William senior to apoplexy, and while

previously financially stable, the death left them in monetary turmoil. Despite this lack of

familial funding, Burnside's mother was able to petition for a scholarship for him, and

her son was able to attain his early schooling at Christ's Hospital School, where he

achieved the highest achievement in the mathematical school (Burnside, Neumann, &

Tompson, p. 89). Upon graduation in 1871, he then attended St. John's College in

Cambridge, and was considered the best man of his year (p. 15). Late in his second year

of undergraduate work, he moved to Pembroke College, graduating in 1875 (p. 15).  He

was known as an expert oarsman while there, and was extremely fond of and talented at

the sport due to his light weight, spare build, and powerful endurance capabilities

(Forsyth, p. 64). After graduation, this love for rowing then turned into a zest and love for

fishing, a passion he carried for the rest of his life.

Upon graduation he received first prize in Smith's competition examination, a

prestigious competition for mathematicians which was considered a high honor, and

was elected a Fellow of Pembroke, a title which he maintained from 1875 until 1886

(Forsyth, p. 64). He spent the next ten years of his life coaching both mathematics and

rowing, both at Pembroke as well as Emmanuel (1876) and Kings (1877) (Burnside, Neumann, & Tompson, p. 15). He was published for the first time in 1883 (2004).

In 1885, Burnside was offered a position as a professor at Royal Naval College in Greenwich which he accepted, choosing to spend the rest of his teaching career at that location (Burnside, Neumann, & Tompson, p. 95). Several years later, he was offered an administrative position back at his alma mater but refused, perhaps partially due to the relaxed lifestyle his current teaching allowed him (p. 97). His time was devoted to teaching and training naval officers, particularly those advanced in areas of kinematics, kinetics, and hydrodynamics. Said to have a patient, stimulating teaching style, Burnside was well loved and admired by his students, although at times he could be said to be a bit severe in his grading (Forsyth, p. 72). At this point in his career his publications began to take off, and by 1887 he averaged about four papers a year (Burnside, Neumann, & Tompson, p. 15).

In terms of family life, Burnside married Alexandrina Urquhart in 1886, and together they were the parents of two sons and three daughters (Forsyth, p. 73). He received his first major honor upon election as a fellow of the Royal Society in 1893 (Burnside, Neumann, & Tompson, p. 15). In this same year, he also published his first paper on group theory (Burnside & Panton, p. 50). Of special interest in this paper was a proof that, excepting $(A_5)$, no finite simple group exists whose order is the product of four primes (Burnside, Neumann, & Tompson, p. 31). His first paper was later followed by the publication of his first book, *Theory of Groups of Finite Order*. In the year 1900 he received what he considered his most meaningful award of honorary status as a Fellow of Pembroke (p. 91).

The Burnside Problem, one of the propositions for which he is best known, was theorized for the first time in 1902 (Burnside, Neumann, & Tompson, p. 15). This initial proposition was followed closely in 1904 by his proposal of the ($p^a q^b$) theorem (p. 15). In 1906, with what has been deemed "grave and characteristic reluctance," (p. 15), Burnside was elected as President of the London Mathematical Society. He served in this role for two years, in addition to being a member of the council from 1899 until 1917 (p. 16). He republished a second edition of *Theory of Groups of Finite Order* in 1911, a version which contained five additional chapters and over fifty percent more content than the original (p. 16).

Burnside's retirement came in 1919, at age sixty seven, upon which time he moved to the country where he could maintain a more leisurely existence (Forsyth, p. 73). He continued his research work until his death, especially in his final years of life delving into the world of probability and statistics. William Burnside died of cerebral hemorrhage on August 21, 1927, at the age of seventy five (Burnside, Neumann, & Tompson, p. 106). Posthumously, his work *Theory of Probability* was later published (p. 15).

Personally, Burnside was a well-liked, if sometimes severe individual (Forsyth, p. 75). He was known to set high standards both for himself and for his colleagues and students, but was able to maintain sympathy even in this critique (Burnside, Neumann, & Tompson, p. 15). Burnside despised the pomp and circumstance that accompanied many prestigious awards and positions, and preferred to work individually. While many contemporary mathematicians of his day collaborated frequently, Burnside preferred to work privately, and to separate himself from mathematical controversy. According to *The*

*Collected Papers* of William Burnside (p. 15), "Although from 1895 onwards Burnside

seems to have kept himself well informed about the published literature of group theory,

he does not appear to have had extensive direct contacts with other mathematicians

interested in the subject." Some notable exceptions appear, but as a general rule Burnside

seemingly preferred to work alone. One of the primary aspects that set his work apart

from many of his peers was the elegance of his writing style (p. 15). His mathematics

were characterized by clear, precise thinking throughout the entirety of a developed issue,

as well as a faculty of lucid expression throughout the argument (p. 23). As a rule, he was

a very concise writer, a fact which set him apart both as a teacher and as a mathematician.

### Group Theory Foundation

One of the most significant building blocks for Burnside's prominence in the

mathematical community was his foundational work in group theory (Forsyth, p. 65).

When Burnside came to the forefront, the study of group theory was relatively new, and

had previously been focused almost exclusively on the study of finite permutation groups

(Burnside, Neumann, & Tompson, p. 45). The origin of the study of groups can probably

be traced back to Galois and his work *Second Memoire* in 1846 (p. 16). Thus, Burnside

first became interested in group theory while the subject matter was less than fifty years

in existence (p. 16).

While the work of Galois began the train of thought, little had been considered in

terms of axiomatization, or the abstraction of a theory. Almost all papers up through

Burnside's time assumed that groups were fundamentally finite (Burnside, Neumann, &

Tompson, p. 17). There were two main aspects regarding the study of this term, the first

being the theory of finite permutation groups and the second being a much more general

subject area of groups. Burnside capitalized on this lack of study, and was passionately eager to draw more mathematicians into this area of work. He expresses this excitement in the preface to one of his later works:

> The subject is one which has hitherto attracted but little attention in this country: it will afford me much satisfaction if, by means of this book, I shall succeed in arousing interest among English mathematicians in a branch of pure mathematics which becomes the more fascinating the more it is studied.
>
> (p. 32)

Passionate to bring attention to abstract group theory, Burnside devoted more than forty percent of his life's work and ambition to this area of mathematics and finite group theory (p. 16). When he later retired from his position in the London Mathematical Society, in his resignation speech he expressed a deep regret at having failed to generate more interest in the topic (p. 32). Nonetheless, his work and ideas in abstract group theory were a catalyst for development in modern work and deserve recognition as being fundamental for current and future thought in this area.

There were four main problems to be considered facing group theory at the turn of the twentieth century (Burnside, Neumann, & Tompson, p. 45). First, theorists had to determine whether a finite simple group of composite odd order existed. Second, the idea that every simple group could be generated by two elements had to be examined. Next came the topic of Burnside's problem, as well as the proof of the $(p^a q^b)$ theorem.

**Develop Foundational Definitions**

In order to consider Burnside's works, one must first have an understanding of fundamental group theory definitions. To begin, consider Burnside's definition of a group (Burnside, Neumann, & Tompson, p. 45):

A system G of h elements of any kind, $\theta_1$, $\theta_2$, ...$\theta_h$ is called a group of degree h if it satisfies the following conditions:

1.) By some prescription, which will be written as composition or multiplication, one may derive from two elements of the system a third element of the same system. In symbols:

$\theta_r \theta_s = \theta_k$

2.) Always $(\theta r\theta s)\theta k = \theta r(\theta s\theta k) = \theta r\theta s\theta t$

3.) From $\theta\theta r = \theta\theta s$ and from $\theta r\theta = \theta s\theta$ follows $\theta r = \theta s$

His definition here, in more modern terms, defines a group as a set with a binary operation that is both associative and includes both identity and inverses.

The next concept to consider, then, is what it means to have a finitely generated group. To do that, one must first understand the definition of a cyclic group, which is informally defined as a group containing some element that can generate the entire group. More formally, the term cyclic group can be defined as:

If G is a group and a is an element of G, then the cyclic subgroup of G generated by a, denoted by $< a >$, is

$$< a > = \{a^n : n \in Z \} = \{ \text{ all powers of a } \}$$

A group G is called *cyclic* if there exists a $\in$ G with G $= < a >$, in which case a is called a *generator* of G (Rotman, p. 29).

In simplest definition, then, a finite group G is one which contains a finite number of elements. The number of elements in the finite group, otherwise known as the cardinality, is also called the order of G, and is denoted jGj (Sprano, p. 23).

Now consider the formal definition of a finite group: If a group G is finite, then for all a in G, there exists t in $Z^+$ such that $a^t = e$, where e denotes the identity element (Sprano, p. 24). It is then useful to define a periodic group, also known as a torsion group (Sahoo & Sury, p. 34), which is a group in which each element has finite order (Fraleigh, p. 334).

An understanding of abelian groups, free abelian groups, rank, orbits, and basis will also be crucial to understanding Burnside's premises. To begin, a group G is abelian if the binary operation associated with it is commutative (Rotman, p. 39). Next, one must understand the formal definition of a free abelian group:

An abelian group having a generating set X, where X is a subset of a

nonzero abelian group G, which satisfies the following conditions

1.) Each nonzero element a in G can be expressed uniquely (up to order of

summands) in the form $a = n_1x_1 + n_2x_2 \ldots n_rx_r = 0$ for $n_i \neq 0$ in Z

and distinct $x_i$ in X

2.) X generates G, and $n_1x_1 + n_2x_2 \ldots n_rx_r = 0$ for $n_i$ in Z and distinct $x_i$ in X

iff $n_1 = n_2 = \ldots = n_r = 0$ is called a free abelian group, and X is

known as the basis for the group (Fraleigh, p. 334).

Knowing now what a free abelian group entails, one can come to the definition of rank: "If G is a free abelian group, the rank of G is the number of elements in a basis of G" (p. 336).

Next consider the term orbit. Orbits of σ, where σ denotes a permutation of a set, are the equivalence classes of that set determined by the equivalence relation: for a, b elements of the set, a ~ b iff b = σ $^n$(a) for some n in the integers (Fraleigh, p. 87). Another related concept that must be considered when working in group theory is the definition of a ring (Rotman, p. 81).

A ring R is a set with the binary operations of addition and multiplication, where:

(1)     R is an abelian group under addition

(2)     a(bc) = (ab)c for all a, b, c in R

(3)     There exists 1 element of R where for all a in R, 1*a = a = a*1

(4)     Distributivity: for all a, b, c in R, a(b + c) = ab + ac and

        (b + c)a = ba + ca

In order to look at some of Burnside's later work, a brief look into matrix notation is also required. As is generally understood, for any field F, M(n, F ) is the set of all nXn matrices over F and GL(n, F ) is said to be the group of invertible matrices among those (Sahoo & Sury, p. 35). The term unipotent denotes a matrix g in GL(n, F ) such that all of its eigenvalues are 1 (p. 36). The trace of a matrix, denoted tr(), is the sum of the elements in the main diagonal of the matrix.  While other definitions may come into play further into the mathematics considered, for now this gives the reader a basis for beginning to look at Burnside's work.

**The Burnside Problem**

In 1902, Burnside's long debated problem was first published. His work was the beginning of research in the subject of combinatorial group theory, which studies the

properties of groups resulting from restraints to their bounds (Fraleigh, 2003). Burnside

asked the question whether or not "a finitely generated group G of finite exponent m, that

is, $x^m = 1$ for all x in G, must be finite" (Rotman, p. 291). A second form of this question,

known as the Generalized Burnside Problem, asks whether any periodic group, free of

constraints on the orders of elements, must therefore be locally finite (Adian, p. 6).

The General Burnside Problem was later proven false. However, Burnside was

initially able to give at least a partial positive answer to this problem in the case of $n = 3$

(Adian, p. 807). Then, in 1940, Sanov produced proof of such a result for $n = 4$, and in

1958 he proved a similar result for $n = 6$ (p. 807).

The first hint of a negative solution was announced by Novikov in 1968 (Hudec,

p. 12). While he was the first to propose the negative of the problem, his early work came

across severe difficulties, and was not fully developed until his work on a jointly written

paper several years later. The first proven negative solution to the problem was originally

produced in 1964 by E. S. Golod (Adian, p. 12). In this, he was able to show that "there

exist infinite 2-generated periodic groups with unbounded periods of elements" (p. 12).

Then in 1968 Novikov, along with several colleagues, produced a series of joint papers,

in which it was proven that: "For any odd period $n \geq 4381$ and any number $m \geq 2$ of

generators, the free periodic group B(m, n) is infinite" (p. 806). The authors used a new

theory which based the majority of the theory's statements using simultaneous induction

on a natural parameter. The essence of this theory was in the classification of periodic

words of given odd period, and their transformations on the basis of periodic relations (p.

12). Upon further research, this theory was eventually improved up to generalizing odd

periods with $n \geq 665$ (p. 807). However, as seen in the next section, counterexamples

were soon found disproving the General Burnside Problem.

**Counter Example to the General Burnside Problem**

In 1964 mathematicians by the names of Golod and Shafarevich came up with the

first counterexample to the General Burnside Problem (*The General Burnside Problem*,

p. 2). To clarify, the General Burnside Problem asks the question: "If G is a finitely

generated group and every element of G has finite order, then must G be finite?" (p. 2).

Two counterexample arguments found in *The General Burnside Problem* will now be

briefly sketched. Both stem from an understanding of the Golod-Shafarevich theorem, the

result of which is as follows: "For F a field, $T = F<x_1, x_2, \dots , x_d>$, a free non-

commutative algebra generated by $x_1, x_2, \dots x_d$, the quotient algebra T/I is infinite

dimensional over F if the coefficients in the power series expansion of $(1 - dz + \Sigma^P_{i=2} r_1$

$z^i) - 1$ are nonnegative" (p. 3). In simpler terms, this theorem states that the quotient

algebra T/I is infinite dimensional over F if the coefficients in the power series expansion

are non-negative (p.3).

The first way of coming to a counterexample to the General Burnside Problem is

by defining the problem in terms of ring theory (known as Kurosh's theory): "If A is a

finitely generated algebra over a field F and every element of A is nilpotent, then must A

be nilpotent?" (*The General Burnside Problem*, p. 4). Recall the definition of nilpotent:

An element a in A is nilpotent if there exists some n such that $a^n = 0$, whereas the entire

group A is nilpotent if there exists m such that $a_1 a_2 \dots a_m = 0$ for all $a_1, a_2 \dots a_m$ in A,

(Fraleigh, p. 176). Based on this understanding, a counterexample can be reached

negating Kurosh's problem (and hence negating Burnside's general problem):

Let $T = F{<}x_1, x_2, x_3{>}$ be the free algebra over the countable field F.

Denote the ideal of T as T', where T' is made up of all the elements of T except

the constant terms, written $T' = t_1, t_2, ...$ Choose some $m_1 \geq 2$, and construct $t_1{}^{m_1}$

$= t_{1,2} + t_{1,3} + ... + t_{1,k}$ where each $t_{1,j}$ in $T_j$. Choose $m_2$ to be large enough so that

$t^m{}_2{}^2 = t_{2,k1+1} + t_{2,k1+2} + ... + t_{2,k2}$ for some $k_2 > k_1$. Continue on in this

construction for sufficiently large powers of $t_3, t_4, ...$ Using this construction, let

I be the ideal generated by $t_{i,j}$ defined above. Consider T'/I. The way that I has

been set up guarantees that each element T'/I is nilpotent. The Golod-

Shafarevich theorem, however, concludes that T'/I is infinite dimensional over

F, and therefore not nilpotent. Thus T'/I serves as a counterexample to Kurosh's

problem. (*The General Burnside Problem*, p. 5)

A second example for the General Burnside Problem specifically is now considered. This

counterexample stems from the original definition of the general problem, and proceeds

through the example as follows:

Begin by constructing a group G satisfying the Burnside Problem

conditions. So let p be a prime number, and let F be a countable field with p

elements. Let $T = F < x_1, x_2, x_3 >$ be the free algebra over F. T* is the ideal

of T consisting of all of the elements of T but not including the constant

term. Then I is the ideal, $t_{i;j}$. Now let $a_1, a_2, a_3$ be elements $x_1 + I, x_2 + I, x_3$

+ I of the quotient T =I. Then set G equal to the multiplicative semi-group

in T =I generated by $1 + a_1, 1 + a_2, 1 + a_3$. Therefore each element of G has

the form $1 + a$ for some a in T =I. Thus the element a is nilpotent by

construction of T*/I implies for sufficiently large n, $a^{pn} = 1$. Since the

characteristic being dealt with is $p(1 + a)^{pn} = 1 + a^{pn} = 1$ implies $(1 + a)$ has

an inverse which implies G is a group. Every element $(1 + a)$ of G then has

finite order which is a power of p, hence G satisfies the given conditions.

Now show that G is infinite. Begin by assuming that G is finite (and

show that this leads to a contradiction). If G is finite, that implies linear

combinations of the elements of G form a finite dimensional algebra B over

F. Since 1, $1 + a_i$ in G, the combination $(1 + a_i) - 1 = a_i$ in B. Thus, 1, $a_1$, $a_2$,

$a_3$ in B. But 1, $a_1$, $a_2$, $a_3$ generate T/I, which is infinite dimensional. By the

Golod-Shafarevich Theorem, the algebra B is also infinite dimensional,

which is a contradiction to assuming that G is finite.

Therefore, G must be infinite, and the second counterexample has been reached

(*The General Burnside Problem,* p. 6).

## Solutions to Generalized Burnside Problem

Next, a worthwhile variety of topics to consider are the special case solutions to the

Generalized Burnside Problem (Rotman, p. 10). While the General Burnside Problem has

been negated, several smaller, simpler examples of this problem can be true, even though

the idea as a whole does not hold. The problem considers a finitely generated group, say

G, that has finite exponent m, (i.e. $x^m = 1$ for all x in G), and whether G must necessarily

be finite (p. 10). To begin construction, one must build a free group of matrices along

with its three respective images.

(1) We define a free group of rank two $F(R[t, t^1])$ made up of 2X2 matrices

with entries in the Laurent polynomial ring $R[t, t^1]$, where R is the

Laurent polynomial ring whose integer coefficients include R =Z

[x, x $^1$, y, y $^1$].

(2) Let the homomorphic image F(R) of F(R[t,t $^1$]) be obtained by putting t =

1 for each element of F. Then F(R) is a group of matrices

isomorphic to the free metabelian group F/F" of rank 2.

(3) A quotient ring exists S = S(n) of R such that for each n 2 $Z^+$, if n is a

prime power, then F(S) is isomorphic, the metabelian free

Burnside group of exponent n.

(4) n in Z is a prime power iff F(S[t,t$^{-1}$]) is solvable (Bachmuth, p. 1).

This manner of construction then gives a commutative square. In this case, the ring

homomorphism R implies S maps horizontally, say α, while the vertical maps are created

by sending the element t to the identity element, say β (Dress, Siebeneicher, & Yoshida,

p. 4). The Burnside map on the free group is then α: F(R) onto F(S) for prime exponents.

Note: on the free group, the Burnside map can be notated β: F(R[t,t $^1$]) onto G for G =

F/F$^q$. The generalized Burnside can be stated as the following: Let γ be a mapping of the

free group F(R[t,t$^{-1}$]) onto a group G such that γ induces α. Then G is the image of γ, and

is solvable (Adian, p. 808)

**Burnside's Restricted Problem**

The Restricted Burnside Problem is yet another byproduct of the original

situation proposed by Burnside. To define this variation of the question: "For

fixed positive integers m and n, are there only finitely many groups generated by

m elements of bounded exponent n? (Shumyatsky & Silva, p. 397). The

restricted problem considers whether there are only finitely many finite groups

with m generators of exponent n, up to an isomorphism. How does this differ

from the bounded Burnside problem? The restricted problem has more

requirements imposed on the structure of the group: not only must there be n

exponents, but this problem considers a strict case of m generators.

  While proofs and examples regarding the Restricted Burnside Problem are not

discussed here due to notational complexities and lack of integrability to the subject

matter as a whole, the results of this case have many practical implications in Jordan and

Lie groups as well as in the study of Engel words (Shumyatsky & Silva, p. 397). In 1990

this problem was proven in the affirmative by Efim Zelmanov, who won the Fields

Medal for this work (Sahoo & Sury, p. 35). His work used both Jordan and Lie group

properties, but later proofs were able to show it using only Lie group identities (Burnside,

Neumann, & Tompson, p. 42). This form of the problem has an advantage in that it can

be looked at using topological techniques.

<div align="center">**Bounded Burnside Problem Definition**</div>

  While Burnside's original problem is beautiful in its forthright proposal, it gives

relatively few restrictions on the structure of the group being considered. Therefore, in

attempting to attack this problem, many mathematicians found it expedient to tweak the

issue some small amount and see what types of results certain alterations to the original

question would produce, in hopes of eventually getting back to the bigger picture. The

Bounded Burnside Problem is one such occasion, with the additional restriction of a set

exponent as defined: "If G is a finitely generated group with exponent n, then is G

necessarily finite?" (Adian, p. 12). This restriction goes back to the idea of periodicity

with exponent n, where there exists a least integer n such that for all g in G, $g^n = 1$.

To truly understand what this refers to, one requires also a definition of what is known as the Free Burnside group:

A free Burnside group is a group with m distinguished generators $x_1$, ... , $x_m$ in which the identity $x^n = 1$ holds for all elements x, and which is the largest group satisfying these requirements. We say this group has rank m and exponent n, and we denote it as B(m, n). (Adian, p. 6)

One of the most important and characteristic properties of the Free Burnside Group is that it maintains uniqueness up to a given isomorphism. Given that G is a group with m generators of exponent n, there exists a unique homomorphism $\phi$: B(m, n) onto G which maps the $i^{th}$ generator of B(m, n) to the $i^{th}$ generator of G (p. 806). These definitions then lead to an alternate, more popular and useful definition of the bounded Burnside problem which asks: For which m, n in $Z^+$ is B(m, n) finite? (p. 806).

In Burnside's original research, he considered two cases. First, he showed that the cyclic group of order n, or B(1,n), was a solution (Burnside, Neumann, & Tompson, p. 19). Second, he showed that the direct product of m copies of the cyclic group of order 2, or B(m, 2) was also true (p. 19). The affirmative result for certain classes of exponents was later shown by Pyotr Novikob Sergei in 1968 (Adian, p. 805). A famous class of counter-examples was also found, known as Tarski Monsters (Hudec, p. 9). This series of counter-examples is finitely generated, non-cyclic infinite groups in which every nontrivial proper subgroup is a finite cyclic group. While the Bounded Burnside Problem did not result in the solution to the general problem that many mathematicians had hoped for, it nevertheless produced interesting and useful results.

**Burnside's Basis Theorem**

The Burnside Basis Theorem states that if G is any finite p-group, then G/ Φ(G) is a vector space over $F_p$, and its dimension is the minimum number of generators of G (Apisa & Klopsch, p. 8). Burnside's Basis Theorem was not as integral to his work in finite group theory, so it is not analyzed in depth here. However, it did have many interesting applications and is worthy of further future review. As such, a brief sketch of the necessary topics to be analyzed will be mentioned here.

In alternate form, this theorem claims that all finite p-groups are also β-groups with the basis property. The basis property refers to a group in which all of its subgroups have the β property. The term β-group is used to describe groups with the β property, otherwise known as the weak basis property. A group is said to have the weak basis property if the size of all of its minimal generating sets is the same (Apisa & Klopsch, p. 8).

Studying this topic must include a brief look at β-groups as defined by Apisa and Klopsch: "A β-group is a group such that all its minimal generating sets (with respect to inclusion) have the same size" (p. 8). Noting that a generating set of a group is considered minimal if no proper subset of it is able to produce the entire group, it can be shown that taking the quotient of every finite β-group again yields a β-group (p. 8). Using that proposition then leads to the theorem that every finite β-group must be solvable. Using that result, as well as a classification of Frattini-free finite β-groups, leads to this eventual conclusion: "Let G be a finite group. Then G is a β-group iff G is a p-group for some prime p" (p. 8). With more formal versions of

these definitions and notational remarks, it can be shown that the theorem holds true

for all finite p-groups.

## Burnside's Ring

Another application of Burnside's work is known as the Burnside ring. First

consider the definition of the Burnside ring, as seen in *Representations*:

For a finite group G, the Burnside ring $\Omega(G)$ of G is defined to be the ring

generated by the formal differences of isomorphism classes of G-sets. The

ring structure is given by disjoint union and Cartesian product of G-sets. This

ring is therefore the Z-module generated by the conjugacy classes of

subgroups of G. (Montaldi, p. 8)

But what exactly does that mean? The definition of a ring was shown previously.

Consider, then, an isomorphism class, a term which simply refers to a collection of items

that are isomorphic to each other. An isomorphism of a set S with a set T is a one to one

function $\Phi$ which maps S onto T such that $\Phi(x* y) = \Phi(x) o \Phi(y)$ for all x,,y in S

(Sprano, p. 30).

The term G-set also requires an explanation. For G a group and S a set, we define

S as the left G-set if there exists some function $\Psi$: (GxS) onto S such that $\Psi(g_1, \Psi(g,s)) = \Psi(g_1 g_2,s)$ for all S in S and $g_1$, $g_2$ in G (Montaldi, p. 4). The map is then called a left G-

action on the set S (Terr, p. 2).  In simpler terms, this can be defined as any set X which a

group G acts on through a function that contains both associativity and conservation of

the identity element (Walcott, p. 1). Considering the next unfamiliar term, a disjoint

union is simply a union of two sets in which the elements are indexed and kept according

to their original set (Rotman, p. 428). The definition of conjugacy class is closely related

to the previously mentioned definition of an orbit:

> A conjugacy class is an orbit of a group (as a set) under the action of the
>
> group on itself by conjucation such that:
>
>   1.) for any x, y in a subset c, there exists g in G such that $g \times g^{1} = y$.
>
>   2.) if x in C and g in G, then $g \times g^{1}$ in C. (Fraleigh, p. 87)

With these definitions in mind, a reader can begin to form the impression of a Burnside

Ring. The existence of this map is the basis for many congruence relations within

elementary finite group theory, and while not one of Burnside's most popular

accomplishments, this map still had significant effects on finite theory.

## Burnside's Lemma

First, a definition of the Burnside Lemma is in order:

> Let K be any field and let G a subset of GL(n, K) be a subgroup such that the
>
> set tr(g): g is finite, of cardinality r. Assume that no nontrivial element of G is
>
> unipotent. Then G must be finite and of cardinality less than or equal to $r^{n2}$.
>
> (Rotman, p. 76)

Next, consider the proof of Burnside's Lemma, as found in (Sahoo & Sury, p. 34-

38):

> Let $\{g_1, ..., g_d\}$ in G.  This forms a basis for the vector subspace of all nXn
>
> matrices over G.  In order to count the elements of G, the ordered d-tuple
>
> $(tr(g_1 g), tr(g_2 g), ..., tr(g_d g))$ is associated with each g in G.  Assuming that
>
> the same of these tuples were associated with two separate elements of G,
>
> say x and y, would lead to $tr(g_i(x\ y)) = 0$ for all i less than or equal to d.  For

any $k \geq 0$, $(i-x^{-1}y)^k x^{-1} = \Sigma^d_{i=1} \lambda_i g_i$ for some $\lambda$ in C. Thus, multiplying the i-

th equation $tr(g_i(x-y)) = 0$ by $\lambda_i$, and adding all of them, we get $tr(i-x^{-1}y)^{k+1})$

$= 0$. Since this holds for all $k \geq 0$, we must have $i-x^{-1}y$ to be a nilpotent

matrix h; that is, all eigenvalues of h are 0. Hence $x^{-1}y$ is i-h, which is clearly

unipotent. This means x=y. Hence the association g mapping to $(tr(g_1g)$,

$tr(g_2g)$, ..., $tr(g_dg))$ is one-to-one. As the traces of elements of G take at the

most r values, the set of d-tuples above has cardinality at the most $r^d \leq r^{n2}$.

## Burnside's $p^a$ $q^b$ Theorem

The Burnside $p^a * q^b$ theorem is as follows:

If G is a nonabelian finite simple group, then 1 is the only conjugacy class

whose size is a prime power. Therefore, every group of order $p^m * q^n$, where

p and q are primes, is a solvable group. (Rotman, p. 192).

Specific definitions of characters, trivial characters, representations, and

irreducible representations must first be considered.

To begin, a character of a group G in a field F is a homomorphism $\varphi$: G onto $F^X$,

where $F^X$ is the group of multiplicative nonzero elements in F (Rotman, p. 193). Beyond

that, a trivial character, denoted $\chi_i$ is given by the trivial representation $\varphi$: G onto C where

for all g in G, $\varphi(g) = 1$ (p. 568). A representation, then, is a function $\Phi$: G onto GL(G)

where $\Phi(g)$ : h onto gh for all g, h in G (p. 568). Building on that definition, an

irreducible function has as its character the function $\chi_0$: G onto C defined by $\chi_0 (g) = tr($

$\Phi(g))$. Remember, the trace of an n by n matrix is simply the sum of its diagonal entries.

With those definitions in mind, building a proof of this theorem should be

relatively simple (Glauberman, p. 469):

Contrary to the theorem's assumption, assume that $h_j = p^e$ where $p^e > 1$ for some j. Then for all i:

$$Z(G/Ker(\chi_i)) = \{\ g \ in \ G : \| \chi_i (g) \| = n_i \ \}.$$

Let $\theta = \chi\sigma$ be an irreducible character given by a $\sigma$ representation of a finite group G. Since G is given to be a simple group that implies $Ker(\chi_i)$ = {1} for all i. Thus, $Z(G/Ker(\chi_i)) = Z(G) = \{1\}$. By a proposition to Schur's Lemma, (not shown), if $(n_i, h_j) = 1$ for some i, j then $\| \chi_i (g_j) \| = n_i$ or $\chi_i (g_j) = 0$. If $\chi_i$ is the trivial character defined above, then $\chi_i (g_j) = 1$ for all j. If, on the contrary, $\chi_i$ is not the trivial character that implies $\chi_i (g_j) = 0$. However, if $(n_i,h_j) \neq 1$ implies $p \| n_i$ for $h_j = p^e$. Thus, for all i not equal to one, $\chi_i (g_j) = 0$ or $p \| n_i$.

By the orthogonality relation, $\Sigma^{Pr}_{i=1} \ n_i \chi_i (g_j) = 0$ for $n_1 = 1$, $n_1 = \chi_i$ $(g_j)$. Every other $n_i$ can be expressed in the form $p \ \alpha_i$, where $\alpha_i$ is an algebraic integer, or is equal to 0. Therefore, $0 = 1 + p\beta$ for $\beta$ an algebraic integer. However, this would imply that the quotient $-1/p$ is an algebraic integer which implies $-1/p$ is in Z, which is a contradiction. Therefore, the theorem's assumption holds, and every group of order $p^a \ q^b$ is hence solvable.

Burnside's theorem has many interesting applications in counting, and was one of the first of his works to gain him widespread recognition as a mathematician. While Burnside was not the originator of the idea, (that accomplishment went to George Frobenius), he was the first to publish and spread this line of thinking. A second, less known version of the theorem also exists, asserting that

Suppose $\|G\| = p^a * q^b$ for two distinct primes p and q and nonnegative

integers a and b. Assume that $p^a > q^b$. Then $O_p(G) \neq 1$, except possibly in the

following cases:

1.) p = 2 and q is a Fermat prime;

2.) q = 2 and p is a Mersenne prime. (Glauberman, p. 469)

Burnside was able to give examples to prove that the cases in (1) and (2) must be

excluded. In terms of the usefulness of this theorem, under a group of permutations, this

can be used to calculate the number of nonequivalent arrangements of objects in a set.

This, and many other useful counting applications, has made Burnside's $p^a q^b$ theorem

practical for decades, and is still useful today.

## Correspondence with other Mathematicians

Burnside typically worked in solitude, but there were a few distinct exceptions worth

remarking upon. Most notably, two letters of correspondence between Burnside and

Robert Fricke have recently been discovered which mark an interesting acquaintanceship.

The first time that their paths crossed was in Volume 52 of the non-British journal

*Mathematische Annalen* in 1898 (Adelmann & Gerbracht, p. 34). Burnside published a

paper on the simple group of order 504, and only one issue later, Fricke published a

remarkably similar article titled "On a simple group of 504 operations" (p. 36).

While dealing with the same subject matter, the two mathematicians utilized

remarkably different style and content, with Burnside focusing on the algebraic identities

compared to Fricke taking a much more geometric approach. The combination of these

two approaches, however, complement each other greatly, even though the two

mathematicians had never before met. Burnside was described as working "in isolation,

possibly even more so than was normal for his times, with little opportunity (or, perhaps, inclination) to discuss his ideas with others" (Adelmann & Gerbracht, p. 34). When Robert Fricke contacted Burnside in a quest to gain his insight into Poincare's article (research which he needed for his pending book, *Automorphic Functions 2*, Burnside responded quickly, beginning a chain of idea exchanges that proved mutually beneficial. Burnside even went so far as to invite Fricke to visit him in his home, but no evidence has been discovered that such a meeting ever occurred (p. 40).

Fricke's original question dealt with the representability of automorphic functions by Poincare series, and Burnside stated the general impossibility, later sending further proof of such a scenario. The interesting part of the correspondence comes in the third page of the letter, after Burnside provided a fully comprehensive explanation in regards to the issue that Fricke originally put forth. Here it is that the very initial case of the Burnside problem is stated:

> I take the opportunity of asking you, whether the following question has ever presented itself to you; and if it has, whether you have come to any conclusion about it. Can a group, generated by a finite number of operations, and such that the order of every one of its operations is finite and less than an assigned integer, consist of an infinite number of operations?" (Adelmann & Gerbracht, p. 41)

In a second letter to Fricke, Burnside later gives hint to his underlying work leading up to his famous paper, giving a general definition of a Burnside group. Their correspondence seemingly continued over several letters (only two of which have been recovered), but eventually their relationship fizzled and died. Nonetheless, their collaboration and

willingness to guide each other gives insight into Burnside as a mathematician and as a

researcher. As Adelman states,

> When the occasion arose and Burnside came into direct contact with one of its
>
> more exposed members, namely Robert Fricke, who had shown a profound
>
> versatility in those areas close to his heart, he seized the opportunity and
>
> allowed a deeper insight into his own current research." (p. 34)

Perhaps it was not that Burnside despised working with others so much as it was that he

was extremely selective in those that he chose to associate with.

### Statistical Work

In addition to Burnside's correspondence with Fricke, in the later years of his life

he also developed a keen interest in statistics, leading him to reach out to several other

key mathematicians. After retirement from the Naval College, Burnside produced ten

short papers as well as the manuscript for a book in statistical research (Aldrich, p. 51).

While his work in this area was undoubtedly of less import than that of his work in group

theory, his contributions to statistics are also worthy of consideration.

Burnside had extremely limited background in statistics when he first began his

inquiries into this subject area. He states, "I have no proper acquaintance with either the

phraseology or the ideas of the modern theory of statistics" (Aldrich, p. 58). His initial

interest seems to have stemmed from his military background, the result of trying to

reduce a military question into a purely mathematical form. To further his knowledge

base, Burnside began a three year correspondence with Ronald Fisher, who later was

known as the leading statistician of that era. Burnside challenged the result of Student (a

leading figure in the field of statistics), worrying about the assumptions that precede the formula.

Burnside's goal was to create a posterior probability statement, calling the statement that the precision constant coincides with its estimated value a false assumption. Burnside's formula gives a narrower interval for the original, but his writings on this subject were met with great criticism from his peers. Many of his correspondences with Fisher revolved around this idea and similar ones, as Burnside spent some time reviewing *Foundations* (Aldrich, p. 54).

Contrary to Fisher's hopes, however, Burnside was hung up on two lines about the "infinite hypothetical population," rather than the larger vision of the paper. The paper relationship between the two researchers occurred in short bursts over a three year period, but eventually Burnside became frustrated as he waited for a response. The resulting minor argument between the two ended the trail of letters. In terms of significant results, the transactions between them could easily be termed a "great waste of time," as neither of them managed to see eye-to-eye on hardly any of the content they discussed (Aldrich, p. 74). However, much of their correspondence and discussion generated thoughts crucial to the manuscript Burnside worked on up until his death. This book was likened to no other statistical book before it: it considered both combinatorial probability, geometrical probability, and the theory of errors all at once. While this title did not offer a wealth of new information, and could be described as somewhat random in its topic order and organization, it produced a concise representation of probability in the nineteenth century and a close examination of old issues. Even though Burnside's impact

on statistics could never match his importance to group theory, he did play a crucial role in its development and pursuit.

## Conclusion

In conclusion, William Burnside was a foundational force for generating interest in modern finite group theory. His problem, while not solved affirmatively in the way he originally anticipated, has been debated by many a mathematician. While to some his work may seem to have little practical significance, to the observant mathematician it is apparent that Burnside's true impact comes not primarily from the mathematics involved, but rather from the force of interest that his work provoked. His work sparked a key level of interest in the subject matter, and without that spark the field of finite group theory might not exist today. Offshoots of the Burnside problem such as the Restricted Burnside Problem, the Bounded Burnside Problem, the General Burnside Problem, and other specific cases are still areas of debate today, and keep mathematicians searching for ways to improve on his work and expand group theory as a whole. His other work, both in statistics and military strategy, has also earned highest acclaim, and his expertise is re-known in a variety of subject areas. He was published a multitude of times. Known as a teacher, researcher, statistician, husband, peer, and mathematician, he was appreciated by many.

References

Adelmann, C., & Gerbracht, E. H. (2007, September 14). Letters from William Burnside

    to Robert Fricke: automorphic functions, and the emergence of the Burnside

    Problem. *Institute for Analysis and Algebra*, *63*, 33-50.

Adian, S. I. (2010).  The Burnside Problem and related topics.  *Russian Mathematical

    Surveys, 65*(5), 805-855.

Adian, S. I. (2010, November 10). The Burnside Problem on periodic groups and related

    questions. *Steklov Institute of Mathematics*, *272*, 5-28.

Aldrich, J. (2008, March 25). Burnside's engagement with the "Modern Theory of

    Statistics." *University of Southampton*, *63*, 51-79.

Apisa, P., & Klopsch, B. (2013, February 10). A generalization of the Burnside Basis

    Theorem. *Journal of Algebra*, *400*, 8-16.

Burnside, William, Neumann, P., Mann, A., & Tompson, Julia. (Ed.). (2004). *The

    Collected Papers of William Burnside* (Vol. 1). Oxford University Press.

    Retrieved from https://books.google.com/books?id=MOO2rEEBMaEC&qd=the

    +collected+papers+of+william+burnside&source=gbs_navlinks_s.

Burnside, W. S., & Panton, A. W. (1960). *The theory of equations with an introduction to

    the theory of binary algebraic forms* (Vol. 1). New York, NY: Dover

    Publications, Inc.

Dress, A., Siebeneicher, C., & Yoshida, T. (1990). Applications of Burnside rings in

    elementary group theory. Retrieved from http://www.emis.de/journals/SLC/o

    papers/s25dress.pdf

Forsyth, A. Russell. (1918). *Theory of functions of a complex variable* (3d ed.).

Cambridge: University press.

Fraleigh, J. B. (2003). *A first course in abstract algebra* (7th ed.). N.p.: Pearson

Education, Inc.

Glauberman, G. I. (1975, December). On Burnside's other theorem. *Pacific Journal of

Mathematics*, *56*(2), 1-11. Retrieved from

<http://msp.org/pjm/1975/56-2/pjm-v56-n2-p12-p.pdf>

Montaldi, J. (2008, April). Representations: Burnside. In *University Of Manchester:

School of Mathematics*. Retrieved from http://www.maths.manchester.ac.uk/

~jm/wiki/Representations/Burnside

Rotman, J. J. (2010).  *Advanced modern algebra* (2nd ed., Vol. 114). Providence, RI:

American Mathematical Society.

Sahoo, B. K., & Sury, B. (2005, July). What is the Burnside Problem? *Resonance*, 34-48.

Seward, B. (2014, January 23). Burnside's Problem, spanning trees and tilings. *Geometry

and Topology*, *18*, 179-210.

Shumyatsky, P., & Silva, J. C. (2008, May 29). Engel words and the restricted Burnside

Problem. *Monatsh Math*, *159*, 379-405.

Sprano, T. E. (2103). *Abstract algebra*. Lynchburg, VA: Liberty University.

Terr, David. (2016). "G-Set." From *MathWorld*—A Wolfram Web Resource, created by

Eric W. Weisstein. http://mathworld.wolfram.com/G-Set.html

The general Burnside Problem (2009, September 4). In *Uniformly at Random*. Retrieved

February 11, 2016, from https://uniformlyatrandom.wordpress.com/2009/09/04/th

e-general-burnside-problem/

Walcott, K. (2004). *Application and analysis of Burnside's theorem* (Master's thesis). 7

December.