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# Making Waves

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**Abstract**

As explored in [1], the structure of the hyperbolic numbers encodes characteristics of two-dimensional Minkowski spacetime. Differentiable functions on the hyperbolic numbers provide solutions to the wave equation. In this paper we analogize the method of conformal mappings to the hyperbolic numbers in order to provide solutions to the wave equation on given regions with given boundary constraints. As template solutions we use low-degree polynomials and the logarithm function, and as transformations we use polynomials and simple rational functions. We conjecture (and nearly show) that in general, the family of rational functions we use take hyperbolas to hyperbolas.

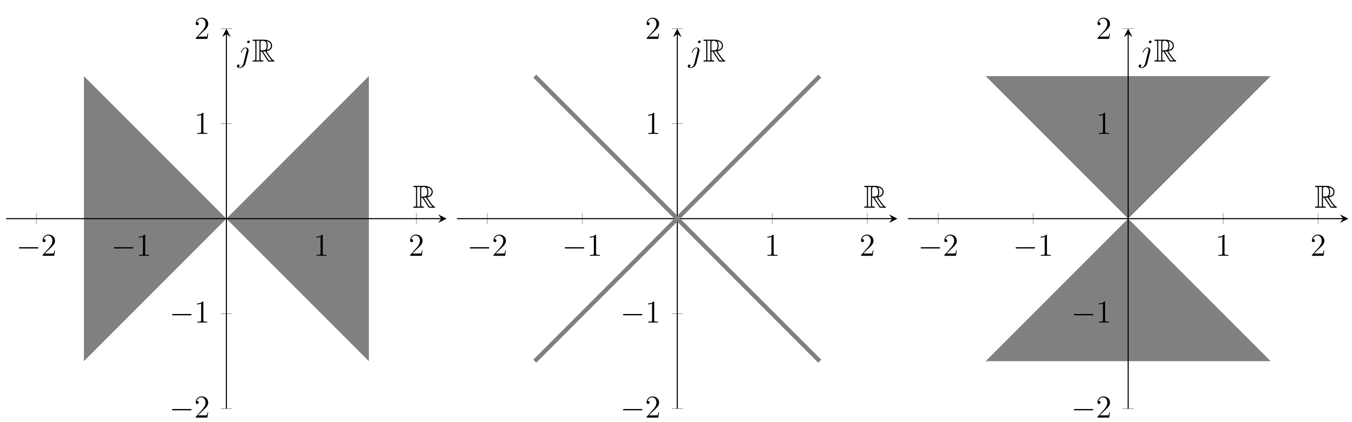
## 1 Introduction

We begin by translating the theory from [1] into our own symbology. We denote the hyperbolic numbers as $\mathcal{H} = \{t + jx : t, x \in \mathbb{R}\}$, where $j^2 = 1$. We define the hyperbolic conjugate of $z = t + jx$ as $\bar{z} = t - jx$, yielding the "squared Minkowski modulus" $\|z\|_M^2 = z\bar{z} = t^2 - x^2$. Then (see Figure 1):

$$\mathcal{H} = \underbrace{\{z \in \mathcal{H} : \|z\|_M^2 > 0\}}_{\text{time-like}} \cup \underbrace{\{z \in \mathcal{H} : \|z\|_M^2 = 0\}}_{\text{light-like}} \cup \underbrace{\{z \in \mathcal{H} : \|z\|_M^2 < 0\}}_{\text{space-like}}.$$

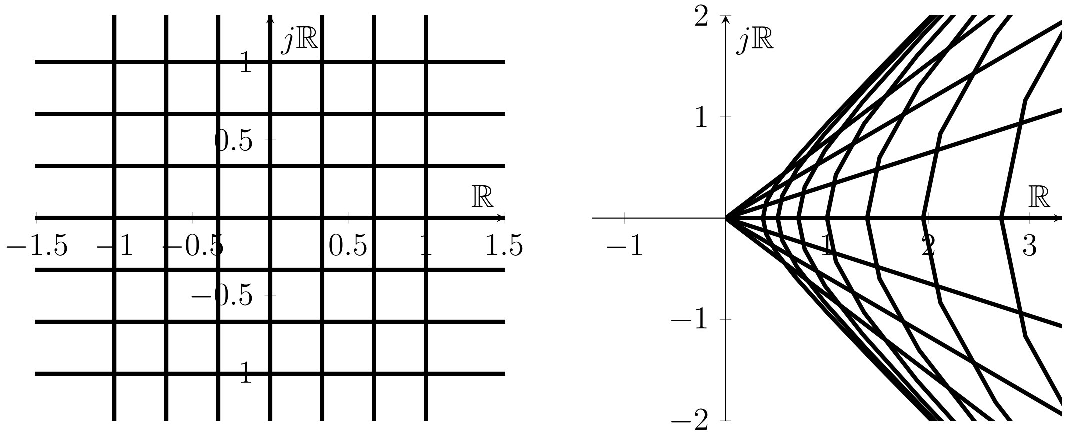
For a given hyperbolic number $z = a + jb \in \mathcal{H}$, we define $L_z : \mathcal{H} \to \mathcal{H}$ for $w = c + jd \in \mathcal{H}$ by

$$L_z(w) = zw = (a + jb)(c + jd) = (ac + bd) + j(ad + bc).$$




(a) time-like (b) light-like (c) space-like

Figure 1: Partition of $\mathcal{H}$ into time-like, light-like, and space-like numbers.




(a) $S = \bigcup_{k=-3}^{3} (\mathbb{R} + jk/3) \cup (k/3 + j\mathbb{R})$

(b) $\exp S$

Figure 2: The hyperbolic exponential function.

We let $\mathcal{R}_{\mathcal{H}} = \{L_z : z \in \mathcal{H}\}$ denote the regular representation of $\mathcal{H}$. It is easy to verify that the elements of $\mathcal{R}_{\mathcal{H}}$ are $\mathbb{R}$-linear transformations on $\mathcal{H}$. This allows us to define the matrix representation of $\mathcal{H}$ as

$$M_{\mathcal{H}} = \{[T] : T \in \mathcal{R}_{\mathcal{H}}\} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a + jb \in \mathcal{H} \right\}.$$

If we consider the "pure hyperbolic number" $z = j\varphi$ for $\varphi \in \mathbb{R}$ then

$$[L_z] = \begin{bmatrix} 0 & \varphi \\ \varphi & 0 \end{bmatrix} \quad \Longrightarrow \quad \exp[L_z] = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & \varphi \\ \varphi & 0 \end{bmatrix}^k = \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}.$$

We define $\exp z$ such that $\exp[L_z] = [L_{\exp z}]$, so $\exp z = \cosh\varphi + j\sinh\varphi$.

# 2 Exponential form

We extend the definition of the exponential to satisfy $(\exp z)(\exp w) = \exp(z + w)$ by defining $\exp(a + jb) = \exp(a)\exp(jb)$ for $a, b \in \mathbb{R}$ and expanding according to the definitions of the real and pure hyperbolic exponential functions. The resulting function is depicted in Figure 2.

Let $z = x + jy \in \mathcal{H}$. Note that

$$\|e^z\|_M^2 = \|(e^x \cosh y) + j(e^x \sinh y)\|_M^2 = (e^x \cosh y)^2 - (e^x \sinh y)^2 = e^{2x}(\cosh^2 y - \sinh^2 y) = e^{2x} > 0$$

so $e^z$ is time-like, and $e^x \cosh y > 0$ so $e^z$ is in the right half-plane; this agrees with our visual data from Figure 2. We now show that $e^{\mathcal{H}} = \{x + jy \in \mathcal{H} : x > |y| > 0\}$; that is, the exponential function is surjective onto this "right quarter-plane". This is easily done by checking that

$$\mathrm{Log}(x + jy) = \ln\|x + jy\|_M + j\tanh^{-1}(y/x)$$

yields $\exp(\mathrm{Log}(z)) = z$ for all $z \in \{x + jy \in \mathcal{H} : x > |y| > 0\}$.

Now once we set aside the zero divisors $\mathbf{zd}(\mathcal{H}) = \{x + jy \in \mathcal{H} : |x| = |y|\}$, we can rewrite our earlier disjoint union of $\mathcal{H}$ as

$$\mathcal{H} = \underbrace{e^{\mathcal{H}} \cup -e^{\mathcal{H}}}_{\text{time-like}} \cup \underbrace{\mathbf{zd}(\mathcal{H})}_{\text{light-like}} \cup \underbrace{je^{\mathcal{H}} \cup -je^{\mathcal{H}}}_{\text{space-like}}.$$

# 3 Properties of conjugate and modulus

The conjugate mapping $z \mapsto \bar{z}$ is obviously a linear transformation if we treat $\mathcal{H}$ as a two-dimensional real vector space, and its composition with itself yields the identity map.

**Lemma 1.** *$\overline{zw} = \bar{z}\bar{w}$ for all $z, w \in \mathcal{H}$.*

*Proof.* Let $z = z_1 + jz_2$ and $w = w_1 + jw_2$ for $z_1, z_2, w_1, w_2 \in \mathbb{R}$. Then

$$\begin{aligned}\overline{zw} &= \overline{(z_1 + jz_2)(w_1 + jw_2)} = \overline{(z_1w_1 + z_2w_2) + j(z_1w_2 + z_2w_1)} \\ &= (z_1w_1 + z_2w_2) - j(z_1w_2 + z_2w_1) = (z_1 - jz_2)(w_1 - jw_2) = (\overline{z_1 + jz_2})(\overline{w_1 + jw_2}) \\ &= \bar{z}\bar{w}.\end{aligned}$$

□

**Lemma 2.** *$\|zw\|_M^2 = \|z\|_M^2\|w\|_M^2$ for all $z, w \in \mathcal{H}$.*

*Proof.* Using Lemma 1,

$$\|zw\|_M^2 = (zw)(\overline{zw}) = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = \|z\|_M^2\|w\|_M^2.$$

□

**Lemma 3.** *$\|1/z\|_M^2 = 1/\|z\|_M^2$ for all $z \in \mathcal{H} \setminus \mathbf{zd}(\mathcal{H})$.*

*Proof.* By Lemma 2,

$$\|z\|_M^2\|1/z\|_M^2 = \|z/z\|_M^2 = \|1\|_M^2 = 1^2 = 1 \quad \Longrightarrow \quad \|1/z\|_M^2 = 1/\|z\|_M^2.$$

□

Thus far we've been using the squared Minkowski modulus; let's now define the nonsquared Minkowski modulus as well.

**Definition 1.** For time-like or light-like $z \in \mathcal{H}$ (that is, $\|z\|_M^2 \geq 0$) we define the Minkowski modulus of $z$ to be $\|z\|_M = \sqrt{\|z\|_M^2}$.

**Lemma 4.** *$\|x\|_M = |x|$ for all $x \in \mathbb{R}$.*

*Proof.* $x = x + j \cdot 0 \in \mathcal{H}$, so $\|x\|_M = \sqrt{x^2 - 0^2} = \sqrt{x^2} = |x|$. □

**Lemma 5.** *$\|zw\|_M = \|z\|_M\|w\|_M$ for all $z, w \in \mathcal{H}$ satisfying $\|z\|_M^2 \geq 0$ and $\|w\|_M^2 \geq 0$.*

*Proof.* Using Lemma 2 and the multiplicativity of nonnegative square roots,

$$\|zw\|_M = \sqrt{\|zw\|_M^2} = \sqrt{\|z\|_M^2\|w\|_M^2} = \sqrt{\|z\|_M^2}\sqrt{\|w\|_M^2} = \|z\|_M\|w\|_M.$$

□

# 4 $\mathcal{H}$-differentiability

As discussed in [1], a hyperbolic-differentiable function $f : \mathcal{H} \to \mathcal{H}$ can be written uniquely as $f = u + jv$ for $u, v : \mathcal{H} \to \mathbb{R}$—that is, $t + jx \mapsto f(t + jx) = u(t + jx) + jv(t + jx)$—where these component functions satisfy $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t}$. If $f$ is twice $\mathcal{H}$-differentiable then we obtain

$$
\begin{aligned}
f_{tt} - f_{xx} &= \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial t}\right] - \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\
&= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t} + j\frac{\partial v}{\partial t}\right] - \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}\right] \\
&= \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial x} + j\frac{\partial u}{\partial x}\right] - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + j\frac{\partial u}{\partial t}\right] \\
&= \frac{\partial^2 v}{\partial t \partial x} + j\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 v}{\partial t \partial x} - j\frac{\partial^2 u}{\partial t \partial x} \\
&= 0,
\end{aligned}
$$

the wave equation. It can also be seen from the above calculation that the component functions of $f$ satisfy the wave equation, in addition to $f$ itself.

As discussed in [1], the chain rule, addition rule, product rule, quotient rule, and integer power rule apply to hyperbolic calculus. For simplicity, we will only be considering functions that are infinitely hyperbolic differentiable. We will term functions of this sort 'hyperbolic smooth' functions. Also, the component functions of these hyperbolic smooth functions, being that they satisfy the wave equation, will be called 'wavy'.

**Example 1.** For $t + jx \in \mathcal{H}$ we have

$$
e^{t+jx} = e^t \cosh x + je^t \sinh x \quad \Longrightarrow \quad J_f = \begin{bmatrix} \partial_t[e^t \cosh x] & \partial_x[e^t \cosh x] \\ \partial_t[e^t \sinh x] & \partial_x[e^t \sinh x] \end{bmatrix} = \begin{bmatrix} e^t \cosh x & e^t \sinh x \\ e^t \sinh x & e^t \cosh x \end{bmatrix} = [L_{e^{t+jx}}]
$$

which satisfies the Cauchy-Riemann equations by inspection, so the exponential function is its own derivative.

# 5 Methodology

Now, the goal of this paper is to determine methods by which one can find solutions to the wave equation on a multitude of regions. Since the composition of two hyperbolic differentiable functions is itself hyperbolic differentiable, hyperbolic differentiable functions work nicely in this pursuit.

Hence, if we can find real valued wavy functions on a handful of regions and map other regions in the hyperbolic plane to these "template regions", then we can produce unique solutions to the wave equation. For example, assume that there is a region $R$ with certain boundary conditions in the hyperbolic plane on which we would like to find a wavy function. Then, instead of attempting to find a real valued wavy function directly, a task that can be rather difficult, we can attempt to find a hyperbolic differentiable function, $\phi(z)$, that will map $R$ to a different region in the hyperbolic plane, $U$.

Our selection of the mapping $\phi(z)$ is determined in such a way as to map the region $R$ to a region $U$ on which we already know a solution to the wave equation. Methods for finding wavy functions on $U$ with given boundary conditions will be discussed in the section devoted to template solutions. The template regions in which we are interested are the regions that coincide with the level curves of the component functions of hyperbolic differentiable functions.

After ascertaining the real valued wavy function on $U$, we can find the hyperbolic differentiable function for which it is a component function. We can call this hyperbolic differentiable function $\psi(z)$. Then, $\psi(z)$ is a hyperbolic differentiable function on $U$ with a real valued wavy component function satisfying the boundary conditions placed on $U$.

Since a function composed of two hyperbolic differentiable functions is itself hyperbolic differentiable, $(\psi \circ \phi)(z)$ is a hyperbolic differentiable function on $R$. Then, with proper consideration to how the boundary of $R$ maps to $U$, we can determine that the respective component function of $(\psi \circ \phi)(z)$ is a real valued solution to the wave equation on $R$.

# 6 Template Solutions

## 6.1 Polynomials

From the product and sum rules it follows that the polynomials are hyperbolic-differentiable. Moving towards utilizing polynomials to find template solutions for the wave equation, we analyze the level curves of polynomials of different degree. We begin by considering polynomials of degree one.

### 6.1.1 Degree One

Let $z = x + jy$ be a hyperbolic variable. Then a hyperbolic polynomial of degree one can be expanded into its component parts as follows:

$$\begin{aligned} f(z) &= (a_1 + ja_2)(x + jy) + (b_1 + jb_2) \\ &= a_1x + a_2y + b_1 + j(a_2x + a_1y + b_2) \end{aligned}$$

From here, it is easy to see that the level curves of the component parts of a polynomial of degree one are lines. Hence, we can utilize polynomials of degree one to find wavy functions that account for boundary conditions on a region formed by two parallel lines. We can demonstrate the usefulness of this fact with an example.

**Example 2.** Consider $y = x + 1$ and $y = x - 1$, two parallel lines in the plane. Moreover, assume that we want to find a wavy function on the region between the two lines that also satisfies some given boundary conditions. Particularly, assume that we want our wavy function, which we will denote as $f(x, y)$, to equal 10 along $y = x + 1$ and equal $-10$ along $y = x - 1$. This is represented graphically in Figure 3.

Then, since any polynomial of degree one can be written as the component function of a hyperbolic smooth function, we need only find a polynomial of degree one satisfying the boundary conditions of the region. To do so, we let $f(x, y) = ax + by + c$ and solve for $a, b,$ and $c$ in such a way as to satisfy the boundary conditions. First, we notice that the slope of both lines is equal to 1, implying that $\frac{a}{b} = 1$. We also need that $f(x, y) = 10$ for all points on $y = x + 1$ and $f(x, y) = -10$ for all points on $y = x - 1$. We can pick two arbitrary points on each line and solve the system of equations. We find the following system of equations:

$$\begin{aligned} \frac{a}{b} &= 1 \\ a(-1) + b(0) + c &= 10 \\ a(0) + b(-1) + c &= -10. \end{aligned}$$

Solving for the system, we find that $a = -10$, $b = 10$, and $c = 0$. Therefore, $f(x, y) = -10x + 10y$ is a wavy function on the region bounded by the lines $y = x + 1$ and $y = x - 1$ that satisfies the stated boundary conditions.

Now, it is obvious that any real polynomial of degree one satisfies the wave equation, being that the second partial derivatives with respect to $x$ and $y$ are equal to zero. However, it is beneficial to identify the

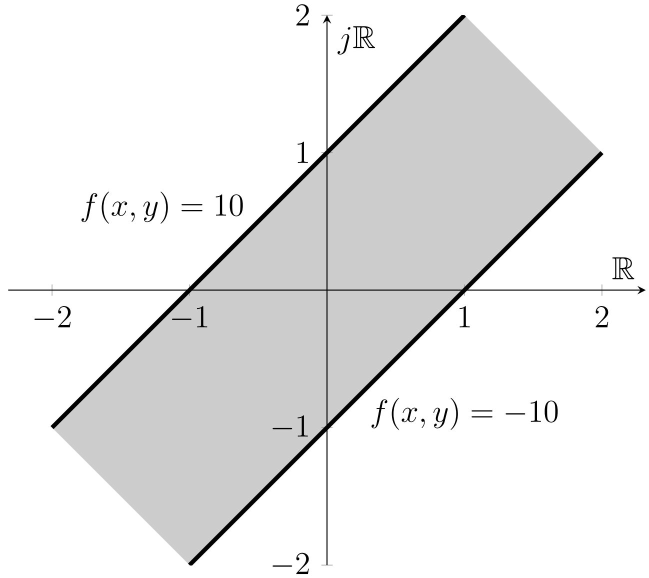



Figure 3: A region between two parallel lines on which we would like to satisfy the wave equation.

first degree hyperbolic polynomial for which the solution above is a component function. The importance of this identification will become evident in our discussion of other solutions to the wave equation.

Using the expansion of an arbitrary first degree hyperbolic polynomial mentioned at the beginning of this section, we find that

$$f(z) = (-10 + j10)z$$

is a hyperbolic polynomial with $f(x, y)$ as a component function.

### 6.1.2 Degree Two

We can now move on to hyperbolic polynomials of degree two. Unlike the polynomials of degree one, it is not the case that every second degree polynomial in x and y satisfies the wave equation. To see this, consider the function $f(x, y) = x^2 + 2y^2$. Notice that,

$$f_{xx}[x^2 + 2y^2] = 2$$
$$f_{yy}[x^2 + 2y^2] = 4.$$

Therefore, $f(x, y)$ is not a solution to the wave equation. However, there are of course other polynomials that do in fact offer a solution to the wave equation. Consider, for example, the function $f(x, y) = x^2 + y^2$. For this reason, we will determine explicitly which second degree polynomials in x and y are wavy.

Begin by considering that a second degree polynomial in x and y is of the form:

$$f(x, y) = a_1x^2 + a_2y^2 + a_3x^2y^2 + a_4x^2y + a_5xy^2 + a_6xy + a_7x + a_8y + a_9$$

for $a_1, ...a_9 \in \mathbb{R}$. Then, taking the second partial derivatives of this function with respect to x and y we find,

$$f_{xx}[f(x, y)] = 2a_1 + 2a_3y^2 + 2a_4y$$
$$f_{yy}[f(x, y)] = 2a_2 + 2a_3x^2 + 2a_5x$$

Therefore, we find that $a_1 = a_2$ and also that $a_3, a_4, a_5 = 0$. From here, we can write the general form of a wavy second degree polynomial in x and y:

$$f(x, y) = a_1(x^2 + y^2) + a_2xy + a_3x + a_4y + a_5$$

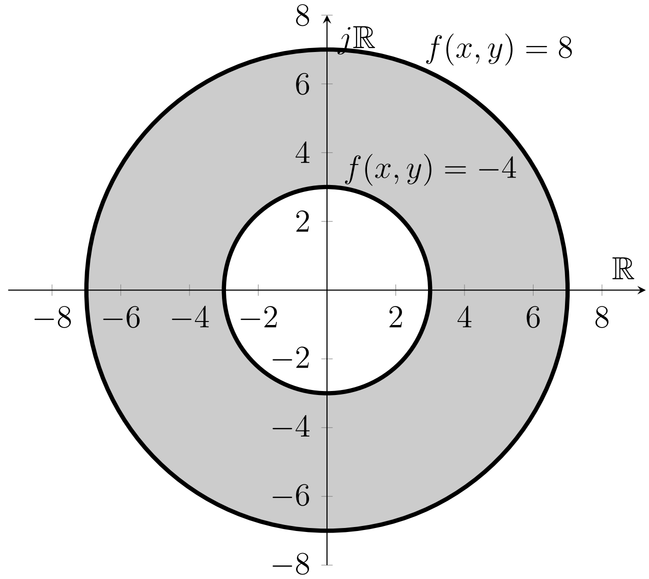



Figure 4: A region between two concentric circles on which we would like to satisfy the wave equation.

*Note: We can expand an arbitrary second degree hyperbolic polynomial and find that the set of component functions of second degree hyperbolic polynomials is equivalent to the set of wavy second degree polynomials in two variables found via the method above. In other words, the only second degree wavy polynomials in two variables are those polynomials that appear as the component function of a second degree hyperbolic polynomial.*

We can see that $f(x, y)$, depending on the coefficient values, will have conic sections as level curves. However, being that the coefficients for $x^2$ and $y^2$ must be equal, we never have a wavy second degree polynomial in x and y with parabolic level curves. Also, note that if we set $a_3$ and $a_4$ equal to zero, the level curves of wavy second degree polynomials in x and y are symmetrical with respect to the lines $y = -x$ and $y = x$. To see this, consider:

$$f(y, x) = a_1(y^2 + x^2) + a_2yx + a_5 = f(x, y)$$
$$f(-y, x) = a_1(y^2 + x^2) - a_2yx + a_5 = f(x, -y)$$

Unfortunately, when $a_3$ and $a_4$ are not equal to zero, we do not always get this nice result. Regardless, without going into needless detail with these polynomials, we can narrow our focus to ellipses and circles that are symmetric with respect to $y = -x$ and $y = x$.

**Example 3.** Second degree polynomials might offer a nice template solution if we are attempting to find a solution to the wave equation satisfying boundary conditions between two concentric circles. For simplicity, and also because the section on transformation mappings will handle translations, we will look at concentric circles centered at the origin.

We attempt to find a second degree polynomial in two variables that is wavy on an annulus centered at the origin and also satisfies given boundary conditions. Let $C_1$ be a circle of radius 3, and let $C_2$ be a circle of radius 7. Assume we want to find a wavy function, $f(x, y)$, such that $f(x, y) = -4$ on $C_1$ and $f(x, y) = 8$ on $C_2$. The region is plotted in Figure 4.

To find such a wavy function. We can set $f(x, y) = a_1(x^2 + y^2) + a_2$. Notice that for $z \in C_1$, $x^2 + y^2 = 9$, and for $z \in C_2$, $x^2 + y^2 = 49$. Then, to find $a_1$ and $a_2$, we solve the following system of equations:

$$9a_1 + a_2 = -4$$
$$49a_1 + a_2 = 8$$

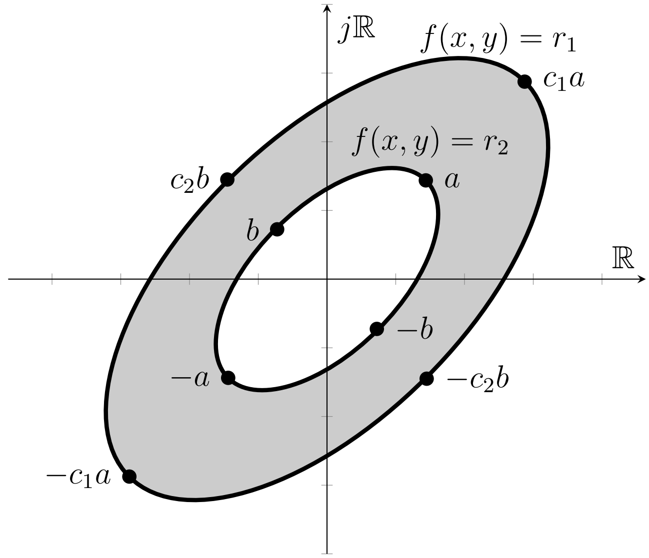



Figure 5: A region between two concentric ellipses on which we would like to satisfy the wave equation.

Solving the system, we find that $a_1 = \frac{3}{10}$ and $a_2 = \frac{-67}{10}$. Moreover, it is obvious that $f(x,y) = \frac{3}{10}(x^2+y^2) - \frac{67}{10}$ is a wavy function being that it is an element of the set of second degree polynomials in two variables that satisfy the wave equation given above. Hence, $f(x,y) = \frac{3}{10}(x^2+y^2) - \frac{67}{10}$ is a solution to the wave equation on the annulus with given boundary conditions.

Lastly, in order to utilize hyperbolic function composition with this template solution, we must find the hyperbolic smooth function for which this real polynomial is a component function. Expanding the general form of a second degree hyperbolic polynomial, we find

$$\begin{aligned} f(z) &= az^2 + bz + c \\ &= (a_1 + ja_2)(x+jy)^2 + (b_1 + jb_2)(x+jy) + c_1 + jc_2 \\ &= (a_1(x^2+y^2) + 2a_2xy + b_1x + b_2y + c_1) + j(a_2(x^2+y^2) + 2a_1xy + b_2x + b_1y + c_2) \end{aligned}$$

Hence, the second degree hyperbolic polynomial in which we are interested is

$$f(z) = \frac{3}{10}z^2 - \frac{67}{10}.$$

**Example 4.** In addition to concentric circles, second degree polynomials provide template solutions on regions bounded by two concentric ellipses that have the major and minor axes on the lines $y = x$ and $y = -x$. Let $E_1$ and $E_2$ be two ellipses centered at the origin with major axes along the line $y = x$ and minor axes along the line $y = -x$. The rest of the pertinent information is given in Figure 5.

For this example, we will keep the constants in Figure 5 arbitrary for the sake of generality. However, $c_1, c_2 > 1$ and $a > b$. To give the equations for both of the ellipses depicted, we use a general equation for an ellipse taken from [2].

The equation of the smaller ellipse is given by

$$\frac{(x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2})^2}{a^2} + \frac{(x\frac{\sqrt{2}}{2} - y\frac{\sqrt{2}}{2})^2}{b^2} = 1$$
$$\implies \left(\frac{1}{2a^2} + \frac{1}{2b^2}\right)(x^2+y^2) + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)xy = 1.$$

and the equation of the larger ellipse is given by

$$\frac{(x\frac{\sqrt{2}}{2}+y\frac{\sqrt{2}}{2})^2}{c_1^2a^2}+\frac{(x\frac{\sqrt{2}}{2}-y\frac{\sqrt{2}}{2})^2}{c_2^2b^2}=1$$
$$\implies \left(\frac{1}{2c_1^2a^2}+\frac{1}{2c_2^2b^2}\right)(x^2+y^2)+\left(\frac{1}{c_1^2a^2}-\frac{1}{c_2^2b^2}\right)xy=1.$$

Then, we want a polynomial $f(x,y)$ in two variables satisfying the wave equation such that $f(x,y)$ at the points on the two ellipses above satisfy the boundary conditions given in Figure 5. Within the equation of the larger ellipse given above, notice that if $c_1=c_2$ then we can rewrite the equation in terms of a constant multiplied by the equation of the smaller ellipse.

Let $c_1=c_2=c$. Then the equation of the larger ellipse becomes

$$\left(\frac{1}{2a^2}+\frac{1}{2b^2}\right)(x^2+y^2)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)xy=c^2.$$

Then, we let

$$f(x,y)=s_1\left[\left(\frac{1}{2a^2}+\frac{1}{2b^2}\right)(x^2+y^2)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)xy\right]+s_2$$

where $s_1, s_2$ are elements of $\mathbb{R}$ such that $f(x,y)$ satisfies the boundary conditions on the two ellipses. To solve for $s_1$ and $s_2$, we simply solve the following system of equations:

$$s_1+s_2=r_2$$
$$s_1*c^2+s_2=r_1$$

and we find that

$$s_1=\frac{r_2-r_1}{(1-c^2)}$$
$$s_2=\frac{r_1-r_2c^2}{(1-c^2)}.$$

Thus, a general form of a wavy polynomial in two variables on the region bounded by concentric ellipses with the given scaling conditions is given by

$$\frac{r_2-r_1}{(1-c^2)}\left[\left(\frac{1}{2a^2}+\frac{1}{2b^2}\right)(x^2+y^2)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)xy\right]+\frac{r_1-r_2c^2}{(1-c^2)}.$$

Then, utilizing the expansion of second degree hyperbolic polynomials, we find that a hyperbolic function with the above equation as a component function is

$$f(z)=\left[\frac{(r_2-r_1)(b^2+a^2)}{(1-c^2)(2a^2b^2)}+j*\frac{(r_2-r_2)(b^2-a^2)}{2(1-c^2)(a^2b^2)}\right]*z^2+\frac{r_1-r_2c^2}{(1-c^2)}.$$

This function is hyperbolic differentiable as well as wavy. Knowing the hyperbolic function will be important when using transformation maps in later sections.

**Example 5.** Lastly, second degree hyperbolic polynomials offer a template solution satisfying the wave equation on the region bounded by two hyperbolas with the same center.

Consider the region given in Figure 6. We wish to find a smooth hyperbolic function satisfying the boundary conditions in the graph. Similar to the previous example, we will use arbitrary constants for the sake of generality throughout this example.

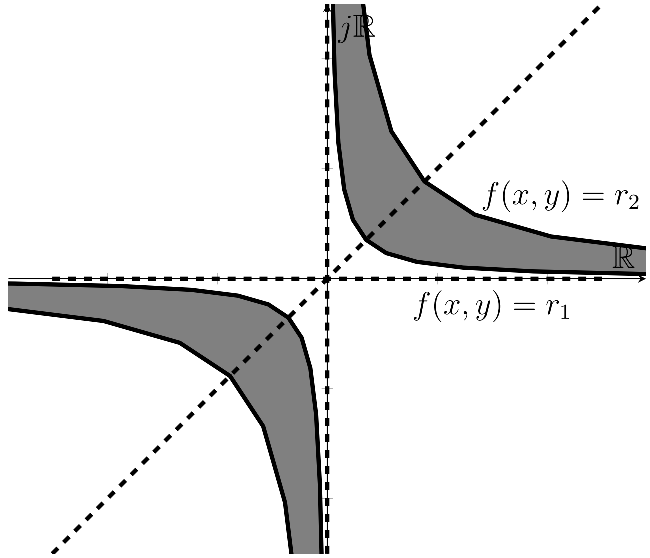



Figure 6: A region between two hyperbolas on which we would like to satisfy the wave equation.

Unfortunately, for this figure, the diagram would likely only become more confusing if the distances from the vertices to the center and transverse axis were marked. However, we will denote the inner hyperbola $H_1$ and denote the outer hyperbola $H_2$. Then, going forth with this example, $a, c_1a$ will be the distances from the center to each vertex respectively, and $b, c_2b$ will be the distances from the transverse axis to each vertex respectively. With these trivialities in place, we can begin with the calculations.

Similar to the ellipse example, we use a the general formula for a rotated hyperbola. The formula for $H_1$ is written as:

$$\frac{(x\frac{\sqrt{2}}{2} - y\frac{\sqrt{2}}{2})^2}{a^2} - \frac{(x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2})^2}{b^2} = 1$$
$$\left(\frac{1}{2a^2} - \frac{1}{2b^2}\right)(x^2 + y^2) - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy = 1$$

and the formula for $H_2$ can be written as:

$$\left(\frac{1}{2c_1^2a^2} - \frac{1}{2c_2^2b^2}\right)(x^2 + y^2) - \left(\frac{1}{c_1^2a^2} + \frac{1}{c_2^2b^2}\right)xy = 1.$$

Again, similar to the ellipse example, if $c_1 = c_2$, then we can set $c = c_1 = c_2$ and rewrite the equation for $H_2$ as:

$$\left(\frac{1}{2a^2} - \frac{1}{2b^2}\right)(x^2 + y^2) - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy = c^2.$$

We can then define a function

$$f(x,y) = s_1\left[\left(\frac{1}{2a^2} - \frac{1}{2b^2}\right)(x^2 + y^2) - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy\right] + s_2$$

and find $s_1, s_2$ so that $f(x,y)$ satisfies the given boundary conditions. We simply solve the following system of equations:

$$s_1 + s_2 = r_1$$
$$s_1 * c^2 + s_2 = r_2.$$

and we find that

$$s_1 = \frac{r_2 - r_1}{c^2 - 1}$$
$$s_2 = \frac{r_1 c^2 - r_2}{c^2 - 1}$$

Thus, a wavy second degree polynomial in two variables is given by

$$f(x, y) = \left(\frac{r_2 - r_1}{c^2 - 1}\right)\left[\left(\frac{1}{2a^2} - \frac{1}{2b^2}\right)(x^2 + y^2) - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy\right] + \left(\frac{r_1 c^2 - r_2}{c^2 - 1}\right).$$

Lastly, we find a hyperbolic second degree polynomial $f(z)$ for which this function is a component function. Using the expanded hyperbolic second degree polynomial, we simply match the coefficients in $f(x, y)$ to the coefficients in $f(z)$ in such a way as to produce $f(x, y)$ upon expansion. We find that

$$f(z) = \left[\frac{(r_2 - r_1)(b^2 - a^2)}{(c^2 - 1)(2a^2 b^2)} - j * \frac{(r_2 - r_1)(b^2 + a^2)}{2(c^2 - 1)a^2 b^2}\right] * z^2 + \frac{r_1 c^2 - r_2}{c^2 - 1}.$$

Then, $f(z)$ is hyperbolic smooth and we are able to compose other hyperbolic smooth functions with $f(z)$ in order to find solutions to the wave equation on other regions in the hyperbolic plane.

Now, at this point it ought to be mentioned that a template solution on regions such as the region given above might be without a purpose, being that mapping another region to these hyperbolas might be difficult or scarcely possible. Regardless, an example of this sort ought to give an idea of how powerful hyperbolic polynomials can be in offering template solutions. Towards the end of the paper, we will provide multiple examples in which we compose transformation mappings and these template solutions to find wavy equations on other regions in the hyperbolic plane.

## 6.2 The logarithm function

Recall from Section 2 that the hyperbolic logarithm is defined on $e^{\mathcal{H}}$ as

$$\text{Log}(x + jy) = \ln \|x + jy\|_M + j \tanh^{-1}(y/x).$$

Note that

$$\ln \|x + jy\|_M = \ln \sqrt{x^2 - y^2} = \frac{1}{2} \ln(x^2 - y^2)$$

and, using equation (3) from [3] since $0 < |y| < x$,

$$\tanh^{-1}(y/x) = \frac{1}{2} \ln\left(\frac{1 + y/x}{1 - y/x}\right).$$

Thus we find that the Jacobian is

$$J_{\text{Log}} = \begin{bmatrix} x/(x^2 - y^2) & y/(y^2 - x^2) \\ y/(y^2 - x^2) & x/(x^2 - y^2) \end{bmatrix} = [L_{1/(x+jy)}]$$

since

$$\frac{1}{x + jy} = \frac{x - jy}{x^2 - y^2}.$$

This shows that the hyperbolic logarithm is differentiable. Since it serves as the inverse of the exponential function, the level curves of its components are precisely the images of horizontal and vertical lines under the exponential function; see Figure 2.

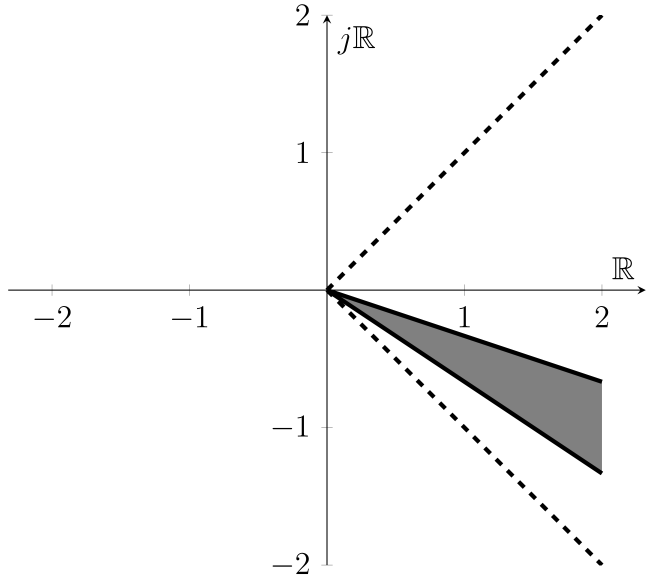



Figure 7: A region in $e^{\mathcal{H}}$ between two rays on which we would like to satisfy the wave equation.

**Example 6.** Let's say that we want to find a wavy function on a wedge in quadrant 4 which is equal to $-10$ on the ray of slope $-\tan(\pi/12)$ and 10 on the ray of slope $-\tan(\pi/6)$; see Figure 7.

The upper line is $y/x = -\tan(\pi/12)$ and the lower line is $y/x = -\tan(\pi/6)$. This gives us

$$\alpha = \tanh^{-1}(y/x) = \tanh^{-1}(-\tan(\pi/12)) \approx -0.3 \quad \text{and} \quad \beta = \tanh^{-1}(y/x) = \tanh^{-1}(-\tan(\pi/6)) \approx -0.7,$$

respectively. These are ugly constants, but they're constants nonetheless. Then if we set

$$a = 20 \cdot \frac{1}{\beta - \alpha} \quad \text{and} \quad b = 10 \cdot \frac{\alpha + \beta}{\alpha - \beta},$$

we get $a \tanh^{-1}(y/x) + b = -10$ on the upper ray and $a \tanh^{-1}(y/x) + b = 10$ on the lower ray. Thus we can take the real component of

$$ja \operatorname{Log}(x + jy) + b$$

as our desired wavy function.

Now for the other level curves.

**Example 7.** Let's say we want to find a wavy function between two "nested" hyperbolas $\frac{1}{2}e^{j\mathbb{R}}$ and $\frac{3}{2}e^{j\mathbb{R}}$ in the right half-plane (see Figure 8), with fixed values on each of those hyperbolas. To continue our exciting precedent, let's say $-10$ on the hyperbola closer to the origin, and 10 on the hyperbola further from the origin.

Note that for each $z \in e^{j\mathbb{R}}$ we have $z = e^{jt}$ for some $t \in \mathbb{R}$, so

$$\|z\|_M = \left\|e^{jt}\right\|_M = \|\cosh t + j \sinh t\|_M = \cosh^2 t - \sinh^2 t = 1.$$

Then looking at our two hyperbolas with Lemmas 4 and 5 yields

$$z \in \frac{1}{2}e^{j\mathbb{R}} \quad \implies \quad \|z\|_M = \left|\frac{1}{2}\right| \left\|e^{jt}\right\|_M = \frac{1}{2} \quad \text{and} \quad z \in \frac{3}{2}e^{j\mathbb{R}} \quad \implies \quad \|z\|_M = \left|\frac{3}{2}\right| \left\|e^{jt}\right\|_M = \frac{3}{2}.$$

Let $\alpha = -\ln 2$ and $\beta = \ln 3 - \alpha$. As in Example 6, if we set

$$a = 20 \cdot \frac{1}{\beta - \alpha} \quad \text{and} \quad b = 10 \cdot \frac{\alpha + \beta}{\alpha - \beta}$$

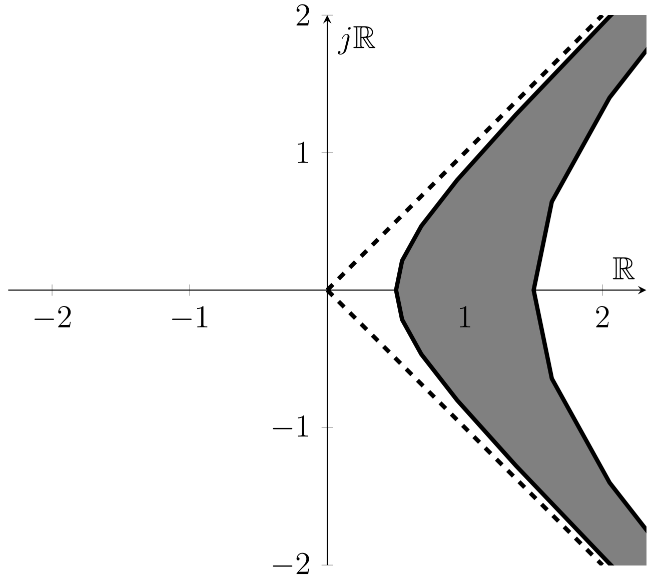



Figure 8: A region in $e^{\mathcal{H}}$ between two hyperbolas on which we would like to satisfy the wave equation.

then we get $a\|x+jy\|_M+b=-10$ on the closer hyperbola and $a\|x+jy\|_M+b=10$ on the further hyperbola. Thus we can take the real component of

$$a\,\mathrm{Log}(x+jy)+b$$

as our desired wavy function.

# 7 Möbiesque® Transformations

Here we develop the theory of the hyperbolic analogue of Möbius transformations: compositions of affine functions $z\mapsto az+b$ and the reciprocal function $z\mapsto z^{-1}$.

## 7.1 Affine

Here we analyze hyperbolic affine transformations geometrically by parametrizing families of affine transformations through a real variable, taking the derivative of these parametrizations, and plotting the resulting vector field.

**Definition 2.** A translation is a mapping $f:\mathcal{H}\to\mathcal{H}$ given by $f(w)=w+z$ for some $z\in\mathcal{H}$.

The simplest type of affine transformation is the translations $z\mapsto z+b$ for some $b\in\mathcal{H}$. This is just familiar vector addition, so we need not spend much time discussing it here.

**Definition 3.** A skew mapping is a mapping $f:\mathcal{H}\to\mathcal{H}$ given by $f(w)=zw$ for some $z\in\mathcal{H}$ such that $\|z\|_M^2\neq 0$.

The more interesting type of affine transformations is those of the form $f(z)=az$ for some $a\in\mathcal{H}$. Let us first consider the case where $a\in e^{\mathcal{H}}$. Then we can write $a=e^c$ for some $c\in\mathcal{H}$, so $a=e^{c_1+jc_2}$ for some $c_1,c_2\in\mathbb{R}$. We can rewrite our function as

$$f(z)=az=e^cz=e^{c_1+jc_2}z=e^{c_1}e^{jc_2}z.$$

If $c_1=c_2=0$ then we simply have the identity transformation. Thus we consider two parametrizations $t\mapsto g_t$ and $t\mapsto h_t$, where $g_t(z)=e^tz$ and $h_t(z)=e^{jt}z$. Note that for a given $c$ as discussed above, we can write any $f$ as $f=g_{c_1}\circ h_{c_2}=h_{c_2}\circ g_{c_1}$.

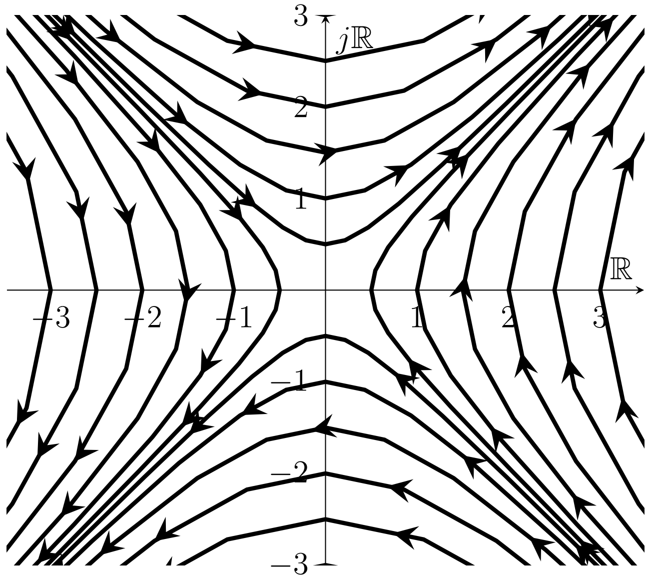
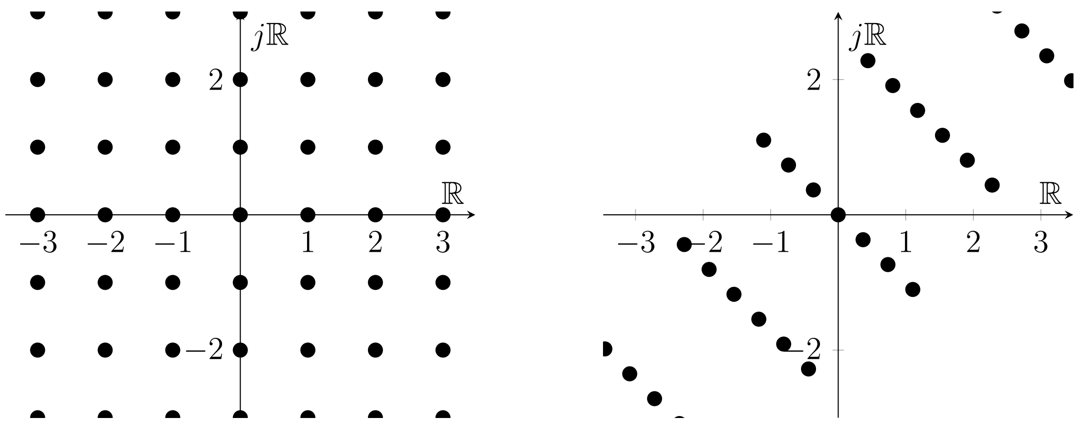



Figure 9: The paths given by multiplication by points in $e^{j\mathbb{R}^+}$.




(a) $\{-3, \dots, 3\}^2$

(b) $e^j\{-3, \dots, 3\}^2$

Figure 10: Multiplication of a lattice by $e^j$.

Our $t \mapsto g_t$ family of functions is trivial; it simply scales each point along a ray from the origin to that point. Thus we will direct our attention to the other family.

Since $g_{t_1+t_2} = g_{t_1} \circ g_{t_2}$ for all $t_1, t_2 \in \mathbb{R}$, plotting the vector field

$$z \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} [g_t(z)] = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} [e^{jt}z] = je^{jt}z\bigg|_{t=0} = jz$$

shows us exactly the path along which points “flow” as we vary $t$; see Figure 9.

Another way to visualize a multiplication by some $z \in e^{j\mathbb{R}}$ is by looking at the image of a lattice under that transformation; see Figure 10. From this we can see that a multiplication by $e^{jc}$ for positive $c$ “stretches” along $y = x$ while “squishing” along $y = -x$, while a multiplication by $e^{jc}$ for negative $c$ does the opposite.

**Definition 4.** If $f$ is a translation and $g$ is a skew, then $f \circ g$ is an affine transformation.

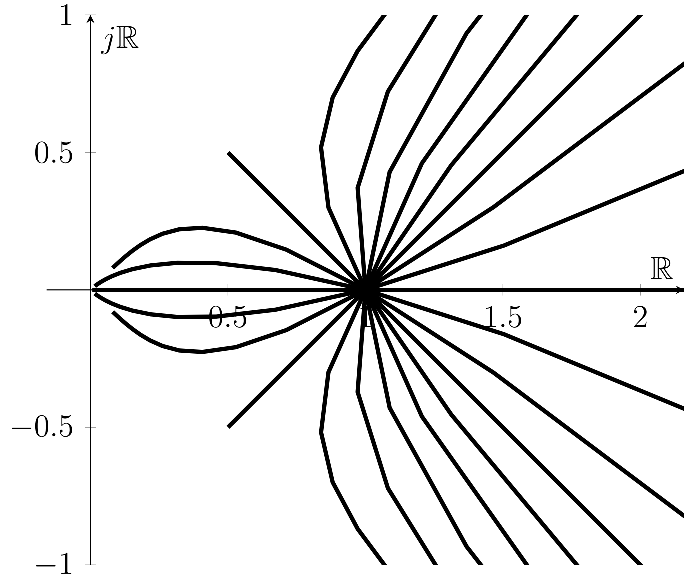



Figure 11: The paths given by the mapping family $e^w \mapsto e^{tw}$ as $t$ goes from 1 to $-1$.

## 7.2 Reciprocal

For non-light-like $z = x + jy \in \mathcal{H}$, the reciprocal of $z$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{\|z\|_M^2} = \frac{x - jy}{x^2 - y^2}.$$

But more interestingly, for $z = e^w \in e^{\mathcal{H}}$ we can express the reciprocal as $z^{-1} = e^{-w}$. This suggests a family of functions similar to the families we studied in section 7.1. We use the parametrization $t \mapsto f_t$ where $f_t : e^{\mathcal{H}} \to e^{\mathcal{H}}$ is given by $f_t(e^w) = e^{tw}$, letting $t$ range between 1 and $-1$. Figure 11 graphs the paths taken by points according to this parametrization.

This figure also exhibits a further geometric quality of the hyperbolic exponential function: it maps the lines $x = y$ and $x = -y$ into the lines $x = 1 + y$ and $x = 1 - y$, respectively.

**Lemma 6.** $e^{\mathbf{zd}(\mathcal{H})} = (1 + \mathbf{zd}(\mathcal{H})) \cap e^{\mathcal{H}}$.

*Proof.* First, for $t \in \mathbb{R}$,

$$\cosh t = \frac{e^t + e^{-t}}{2} = e^{-t} + \frac{e^t - e^{-t}}{2} = e^{-t} + \sinh t.$$

Then for $z = x + jy \in \mathcal{H}$,

$$\begin{aligned} z \in (1+j)\mathcal{H} \quad &\iff \quad z = (1+j)y \\ &\iff \quad e^z = e^y(\cosh y + j \sinh y) \\ &\iff \quad e^z = e^y(e^{-y} + \sinh y + j \sinh y) \\ &\iff \quad e^z = 1 + (1+j)e^y \sinh y \\ &\iff \quad e^z \in 1 + (1+j)\mathcal{H}. \end{aligned}$$

Recall from Section 2 that $\mathbf{zd}(\mathcal{H}) = \{x + jy \in \mathcal{H} : |x| = |y|\}$; thus $\mathbf{zd}(\mathcal{H}) = (1+j)\mathbb{R} \cup (1-j)\mathbb{R}$. By the above (and similar reasoning for $1 - j$), we have

$$e^{(1+j)\mathcal{H}} = (1 + (1+j)\mathcal{H}) \cap e^{\mathcal{H}} \quad \text{and} \quad e^{(1-j)\mathcal{H}} = (1 + (1-j)\mathcal{H}) \cap e^{\mathcal{H}} \quad \implies \quad e^{\mathbf{zd}(\mathcal{H})} = (1 + \mathbf{zd}(\mathcal{H})) \cap e^{\mathcal{H}},$$

completing the proof. □

The above analysis is for the inversion of points in $e^{\mathcal{H}}$, but inversion of points in the other four sections of $\mathcal{H}$ works exactly the same way: you simply follow your unique path through whichever one of $1, j, -1, -j$ you can reach without crossing over a zero divisor, stopping when you reach a point whose modulus is the inverse of your original modulus.

## 7.3 Mapping Hyperbolas to Hyperbolas

**Definition 5.** A Möbiesque transformation is a function

$$f_1 \circ f_2 \cdots \circ f_n,$$

where each $f_k$ is an affine transformation or the reciprocal function.

For our purposes, we only consider hyperbolas whose asymptotes have slope $m$ satisfying $|m| = 1$.

**Definition 6.** A hyperbola is a set of points $H(z, c) = \{w \in \mathcal{H} : \|w - z\|_M^2 = c\}$ for some $z \in \mathcal{H}$ and $c \in \mathbb{R} \setminus \{0\}$.

**Lemma 7.** *For a hyperbola $H$ and translation $f$, the image $f(H)$ is a hyperbola.*

*Proof.* By definition, $H = H(z, c)$ for some $z \in \mathcal{H}$ and $c \in \mathbb{R}$, and $f(w) = w + b$ for some $b \in \mathcal{H}$. Then

$$f(H) = \{w + b : w \in \mathcal{H} \text{ and } \|w - z\|_M^2 = c\} = \{w \in \mathcal{H} : \|w - (z + b)\|_M^2 = c\} = H(z + b, c). \qquad \square$$

**Lemma 8.** *For a hyperbola $H$ and skew mapping $f$, the image $f(H)$ is a hyperbola.*

*Proof.* For any hyperbola $H = H(z, c)$ and skew mapping $f(w) = aw$, we can write $f = t_2 \circ f \circ t_1$ for translations $t_1(w) = w - z$ and $t_2(w) = w + az$. By Lemma 7, translations take hyperbolas to hyperbolas, so without loss of generality we only need consider hyperbolas centered at the origin.

By definition, $f(w) = aw$ for some $a \in \mathcal{H}$ with $\|z\|_M^2 \neq 0$. Then,

$$f(H) = \{aw : w \in \mathcal{H} \text{ and } \|w\|_M^2 = c\} = \{w \in \mathcal{H} : \|w\|_M^2 = c/\|a\|_M^2\} = H(0, c/\|a\|_M^2). \qquad \square$$

The reciprocal function is trickier. It works fine for hyperbolas centered at the origin, but other hyperbolas cross zero divisors which aren't in the domain of the reciprocal function. Thus we'll cheat a bit and extend the definition of image.

**Definition 7.** Let $S \subseteq \mathcal{H}$ and $f : S \to \mathcal{H}$. For a given curve $C \subset \mathcal{H}$, we extend the definition of $f(C)$ to be the closure of $f(C \cap S)$.

Note that this is compatible with our previous lemmas for translations and skew mappings, because hyperbolas are topologically closed.

The above definition should suffice to prove the general version of this lemma, but the proof is evasive.

**Lemma 9.** *For a hyperbola $H = H(0, c)$, the image of $H$ under the reciprocal function is a hyperbola.*

*Proof.* By Lemma 3,

$$f(H) = \{1/w : w \in \mathcal{H} \text{ and } \|w\|_M^2 = c\} = \{w \in \mathcal{H} : \|w\|_M^2 = 1/c\} = H(0, 1/c). \qquad \square$$

The general theorem would follow trivially if that last lemma could be shown for $z \neq 0$:

**Conjecture 1.** *Möbiesque transformations take hyperbolas to hyperbolas.*

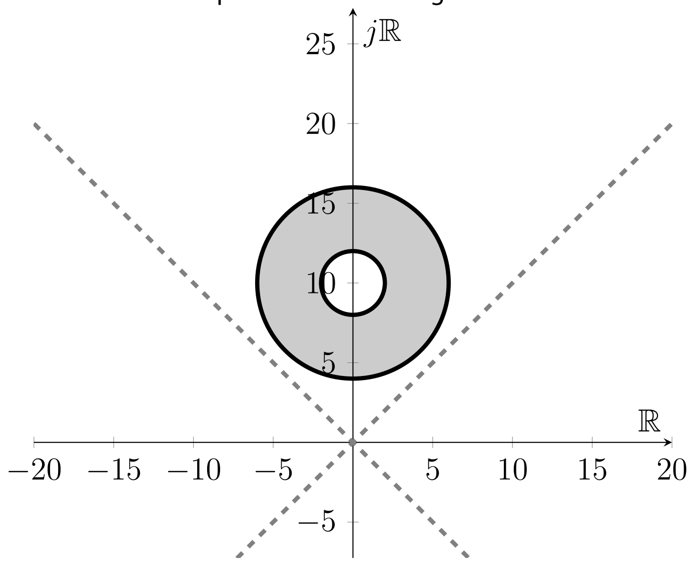



Figure 12: An annulus centered in $je^{\mathbb{H}}$ on which we would like to find a solution to the wave equation.

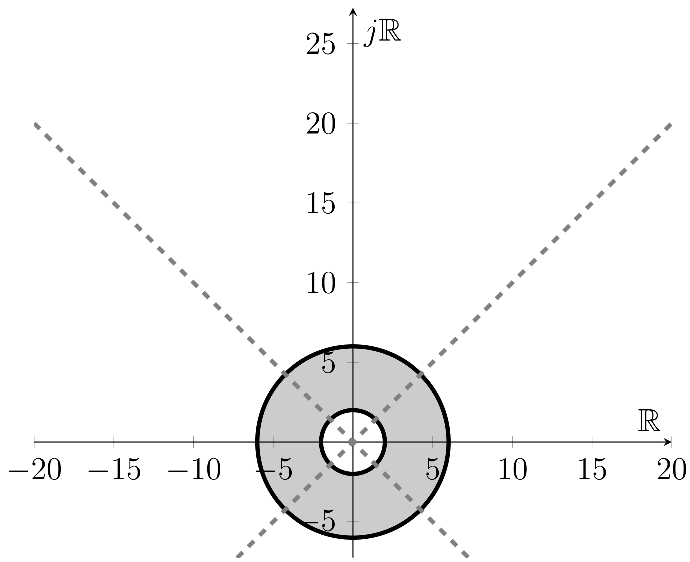



Figure 13: The region $U$; the image of $R$ with respect to $\phi(z)$

# 8 Examples Utilizing Transformations

In this section, the goal is to combine the template solutions that we have found with transformation mappings to find solutions to the wave equation on other regions.

**Example 8.** Assume that we want to find a solution to the wave equation on the region in Figure 12. The annulus in the figure is centered at $z = 10j$ and the radii of the two circles are $r = 2$ and $r = 6$ respectively. Also, although it is not labeled in the figure, assume that we want the solution $g(x, y)$ to satisfy the following boundary conditions. On the outer circle, we would like $g(x, y) = 20$, and on the inner circle, we would like $g(x, y) = -10$. We will call this region $R$.

Now, recall from the template solutions section that hyperbolic polynomials of degree two provide solutions to the wave equation on annuli centered at the origin. Hence, we would like to translate $R$ to an annulus centered at the origin. The function providing this translation is simply $\phi(z) = z - 10j$. The image of $R$ with respect to $\phi(z)$ we will call $U$, which is depicted in Figure 13.

From here, we simply use the method given previously in the paper to find a wavy function for the region $U$. We want $f(x, y) = 20$ on the outer circle of $U$, and $f(x, y) = -10$ on the inner circle of $U$. Then, we let

$f(x,y) = a_1(x^2+y^2) + a_2$ and solve the following system of equations:

$$\begin{aligned} 36a_1 + a_2 &= 20 \\ 4a_1 + a_2 &= -10. \end{aligned}$$

We find that $a_1 = \frac{15}{16}$ and $a_2 = \frac{-55}{4}$. Therefore,

$$f(x,y) = \left(\frac{15}{16}\right)(x^2+y^2) - \frac{55}{4}$$

is a real valued wavy function satisfying the boundary conditions on $U$. Now, by considering the expanded form of a second degree hyperbolic polynomial, we find that $\psi(z)$, the hyperbolic differentiable function with $f(x,y)$ as a component function is

$$\psi(z) = \left(\frac{15}{16}\right)z^2 - \frac{55}{4}.$$

Then, $(\psi \circ \phi)(z)$ is a hyperbolic differentiable function, and so the components functions of $(\psi \circ \phi)$ are wavy on $R$. Therefore, we can expand $(\psi \circ \phi)(z)$ to find a real valued solution to the wave equation on $R$.

Consider,

$$\begin{aligned} (\psi \circ \phi)(z) &= \left(\frac{15}{16}\right)(z-10j)^2 - \frac{55}{4} \\ &= \left(\frac{15}{16}\right)(x+(y-10)j)^2 - \frac{55}{4} \\ &= \left(\frac{15}{16}\right)(x^2+(y-10)^2) - \frac{55}{4} + 2x(y-10)j. \end{aligned}$$

Then, the real valued function satisfying the wave equation on the region $R$ is

$$g(x,y) = \left(\frac{15}{16}\right)(x^2+(y-10)^2) - \frac{55}{4}.$$

To check our solution, note that $x^2+(y-10)^2 = r^2$ is the equation for a circle of radius $r$ centered at $(0,10)$. Hence, it is easy to see that $g(x,y)$ satisfies the boundary conditions. Moreover, just to ensure that $g(x,y)$ is in fact wavy on $R$, we can note that

$$g_{xx}(x,y) = \frac{15}{8} = g_{yy}(x,y).$$

For this particular example, finding the function $g(x,y)$ may have been easy without using the methodology built within this paper. However, for some of the following examples, the reader will find that discovering $g(x,y)$ can be quite difficult.

**Example 9.** Now before we continue with this example, it should be noted that the more complicated examples are nearly always reverse engineered. To find regions on which we can find solutions to the wave equation, we can simply apply invertible transformations to the template regions. Unfortunately, the regions on which the second degree hyperbolic polynomials offer template solutions contain zero divisors. For this reason, using the reciprocal map to map from $R$ to the polynomial template regions is problematic. However, we can use the reciprocal map to map $R$ to a translation of a polynomial template region that does not contain zero divisors.

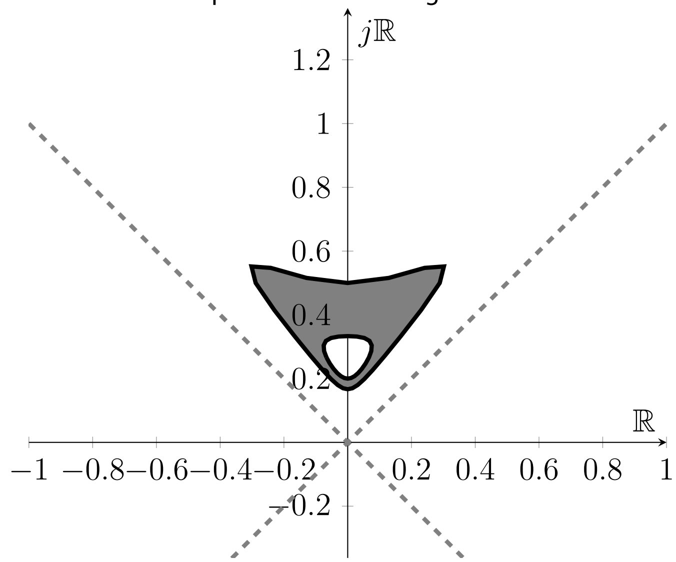



Figure 14: The inversion of an annulus on which we would like to find a solution to the wave equation.

Assume that we want to find a solution to the wave equation on the region $R$ given in Figure 14. The outer curve is given by the following parametrization:

$$x = \frac{2\cos(t)}{4\cos^2(t) - 4\sin^2(t) - 16\sin(t) - 16}$$
$$y = \frac{-2\sin(t) - 4}{4\cos^2(t) - 4\sin^2(t) - 16\sin(t) - 16}$$

where $0 \leq t \leq 2\pi$, and the inner curve is given by the parametrization:

$$x = \frac{\cos(t)}{\cos^2(x) - \sin^2(x) - 8\sin(t) - 16}$$
$$y = \frac{-\cos(t) - 4}{\cos^2(t) - \sin^2(t) - 8\sin(t) - 16}$$

for $0 \leq t \leq 2\pi$. I will leave it to the reader to show that the inversion of this region gives the annulus of inner radius 1 and outer radius 2 centered at $(0, 4)$. Also, we would like the wavy function $g(x, y)$ to satisfy the following boundary conditions:

$$g(x, y) = 5 \text{ on the outer curve,}$$
$$g(x, y) = 2 \text{ on the inner curve.}$$

After applying the reciprocal map to $R$, we get the region $U$ given in Figure 15. We will denote the reciprocal map $\phi$.

Then, just as in the first example, we can translate $U$ to get an annulus centered at the origin. We will call this third region $V$, and it is given in Figure 16. The translation map will be denoted $\gamma(z) = z - 4j$.

Then, we find the wavy function for $V$ via the methods we utilized previously. We find that the wavy function on $V$ is given by

$$f(x, y) = x^2 + y^2 + 1$$

and the corresponding hyperbolic differentiable function is

$$\psi(z) = z^2 + 1.$$

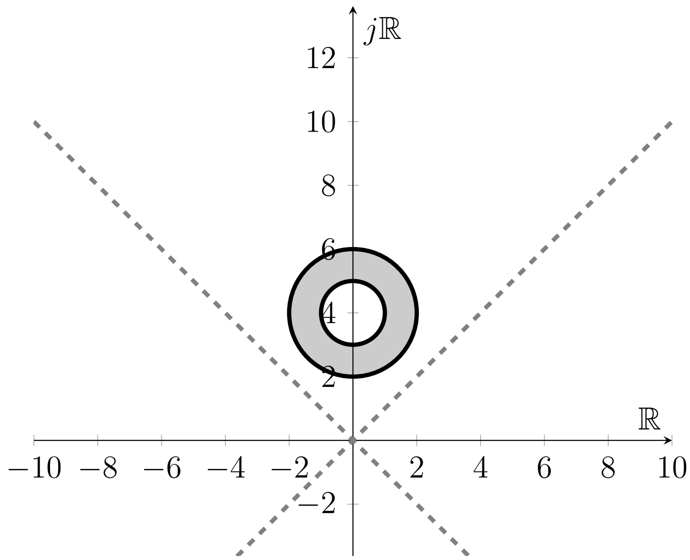



Figure 15: Region $U$; $\phi(R)$.

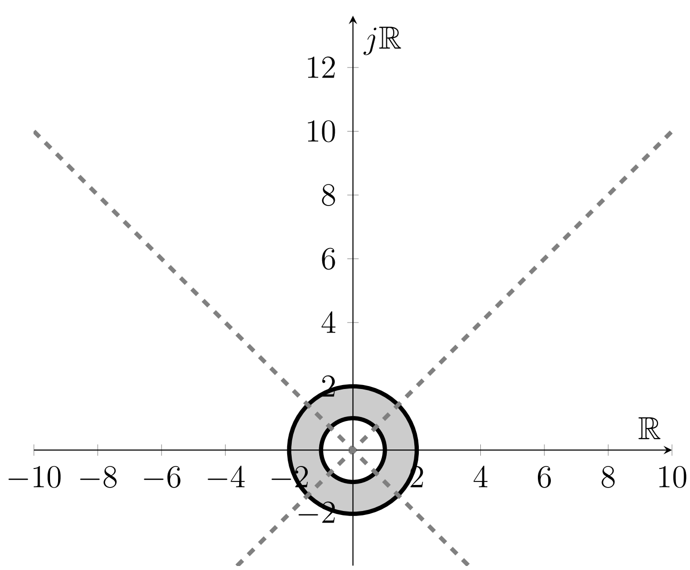



Figure 16: Region $V$; $\gamma(U)$.

From here, we can compose the three pertinent functions to find a solution to the wave equation on the region $R$. Consider,

$$(\psi \circ \gamma \circ \phi)(z) = \left(\frac{1}{z} - 4j\right)^2 + 1.$$

This function is hyperbolic differentiable on $R$. Moreover, expanding to find the component functions, we find that

$$\begin{aligned}(\psi \circ \gamma \circ \phi)(z) &= \left(\frac{1}{x+jy} - 4j\right)^2 - 1 \\ &= \frac{}{(x+jy)^2} - \frac{8}{(x+jy)} * j + 17 \\ &= \frac{1-8y-8xj}{x^2+y^2+2xyj} + 17 \\ &= \frac{(1-8y-8xj)(x^2+y^2-2xyj)}{x^4-2x^2y^2+y^4} + 17 \\ &= \frac{(x^2+y^2-8y(x^2+y^2)+16x^2y)+j(16xy^2-8x(x^2+y^2)-2xy)}{x^4-2x^2y^2+y^4} + 17.\end{aligned}$$

Therefore, the function

$$g(x,y) = \frac{x^2+y^2-8y(x^2+y^2)+16x^2y}{x^4-2x^2y^2+y^4} + 17$$

is a real valued wavy function satisfying the boundary conditions given for $R$. Unlike the previous example, I am bold enough to say that this solution could not be found with a "guess-and-check" methodology. Unfortunately, the more obscure the solution function, the more difficult it is to verify that the function is both wavy and satisfies the boundary conditions. Fortunately, Wolfram Alphra indicates that $g_{xx}(x,y) = g_{yy}(x,y)$ and also that the boundary conditions are met.

**Example 10.** Now, say that we want to find a solution to the wave equation on region $R$ given in Figure 17. In the figure, the left-most ray has a slope of 2 while the right-most ray has a slope of $\frac{6}{5}$. In addition to the figure, we desire the solution function $g(x,y)$ to satisfy the following boundary conditions:

$$\begin{aligned}g(x,y) &= 10 \text{ when } y = 2x \\ g(x,y) &= -10 \text{ when } y = \frac{6}{5}x.\end{aligned}$$

Unlike the template solution, this region is in $je^{\mathbb{H}}$, and hence, we cannot use the hyperbolic logarithm directly.

Instead, we transform the region $R$ to a region $U$ in $e^{\mathbb{H}}$ in order to use the logarithm function. We will use the transformation $\phi(z) = jz$, which reflects $R$ over the line $y = x$. The resulting region $U$ is given in Figure 18. The rays in $U$ have slopes $\frac{1}{2}$ and $\frac{5}{6}$ respectively.

Then, to find a wavy function on $U$, we utilize the method given in the template solution section. We let $f(x,y) = a_1 \tanh^{-1}(y/x) + a_2$ and solve the following system of equations:

$$\begin{aligned}a_1 \tanh^{-1}(1/2) + a_2 &= 10 \\ a_1 \tanh^{-1}(5/6) + a_2 &= -10.\end{aligned}$$

We find $a_1 \approx 30.78$ and $a_2 \approx -6.92$. Thus, a real valued wavy function on $U$ satisfying the boundary conditions is given by

$$f(x,y) = 30.78 \tanh^{-1} -6.92.$$

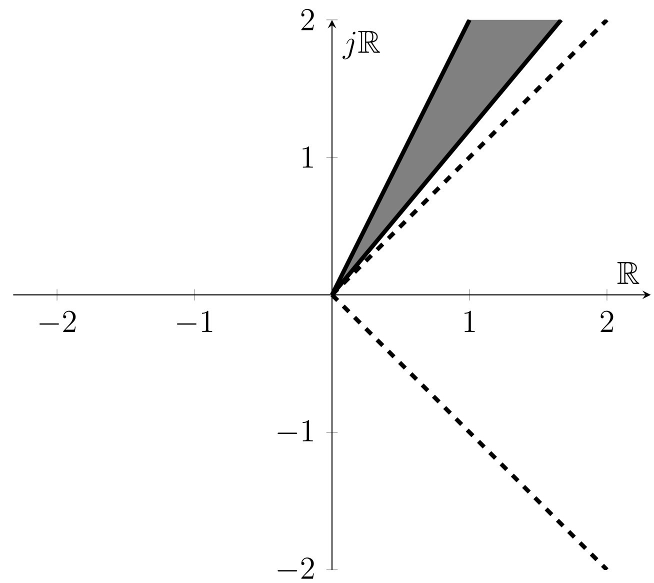



Figure 17: A region in $je^{\mathbb{H}}$ between two rays.

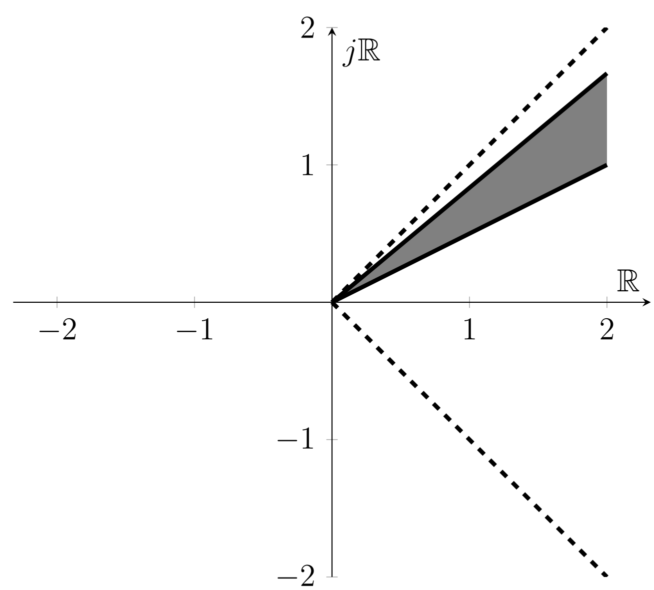



Figure 18: Region $U$, $\phi(R)$.

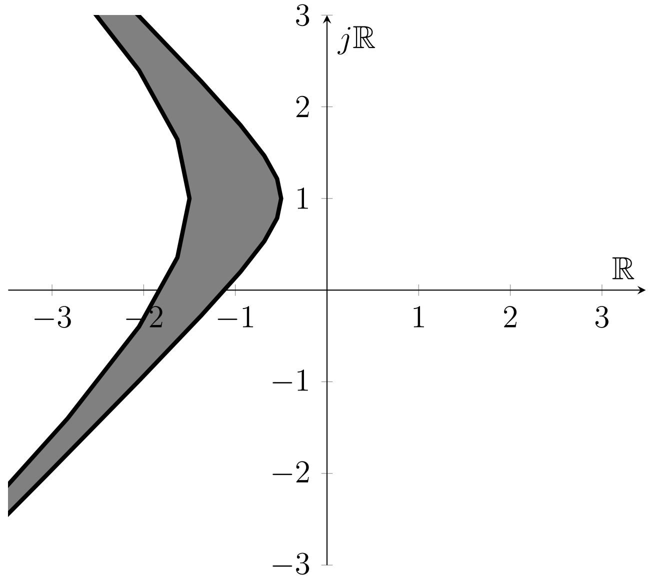



Figure 19: A region in $e^{\mathcal{H}}$ between two hyperbolas on which we would like to satisfy the wave equation.

The hyperbolic differentiable function with $f(x, y)$ as a component function can be written

$$\psi(z) = 30.78j\,\mathrm{Log}(z) - 6.92.$$

Now, we can compose $\phi(z)$ and $\psi(z)$ to find that

$$(\psi \circ \phi)(z) = 30.78j\,\mathrm{Log}(jz) - 6.92.$$

We then expand this hyperbolic differentiable function to find the component function that satisfies the boundary conditions:

$$(\psi \circ \phi)(x + jy) = \frac{30.78}{2}\ln\big(y^2 - x^2\big)j + 30.78\tanh^{-1}(x/y) - 6.92.$$

Hence,

$$g(x, y) = 30.78\tanh^{-1}(x/y) - 6.92$$

is a real valued wavy function on $R$ satisfying the given boundary conditions.

**Example 11.** This example is essentially an extension of Example 7. In this case, our two hyperbolas are $1 - \frac{1}{2}e^{j\mathbb{R}}$ and $1 - \frac{3}{2}e^{j\mathbb{R}}$ in the left half-plane (see Figure 19), again with $-10$ on the hyperbola closer to the origin, and 10 on the hyperbola further from the origin. Let

$$f(x + jy) = a\,\mathrm{Log}(x + jy) + b$$

be the end result from the previous problem. If we compose this with the Möbiesque transformation $g(z) = 1 - z$, then we get back our original regions from before. Thus our full function, taking the real component of the logarithm as described, is

$$h(z) = a\ln\|z\|_M + b \approx 8.04859\ln\|z\|_M - 4.42114.$$

**Example 12.** Let's say we want to solve the wave equation on the area between the two curves pictured in Figure 20, parametrized by

$$e^{\frac{1}{2}\cos\theta}\left[\cosh\left(\frac{1}{2}\sin\theta\right) + j\sinh\left(\frac{1}{2}\sin\theta\right)\right] \quad \text{and} \quad e^{\cos\theta}[\cosh(\sin\theta) + j\sinh(\sin\theta)],$$

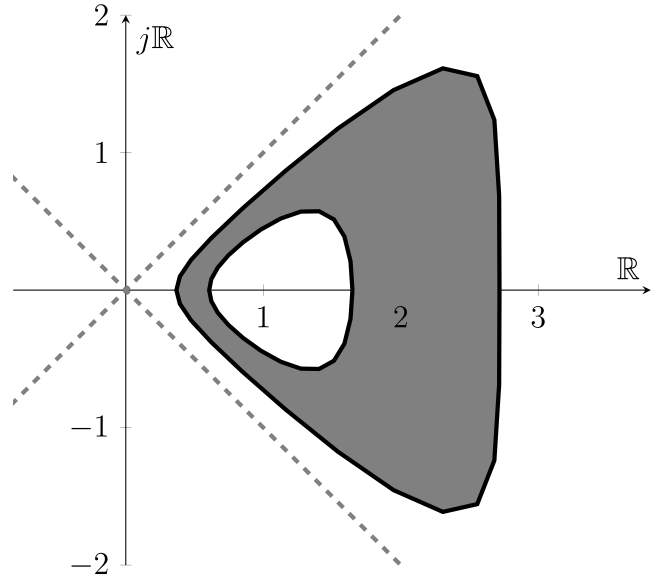



Figure 20: The image of an annulus under the exponential mapping.

yielding −10 on the inner curve and 10 on the outer curve.

The first step is to notice that these curves are the images of the circles of radii 1/2 and 1 under the exponential mapping. Thus we can get back to the original circles using the logarithm function, and then we just need to apply a second-degree polynomial to achieve our desired boundary conditions. Observe that

$$f(x + jy) = (x + jy)^2 = (x^2 + y^2) + 2xyj.$$

This means that for the unit circle, the real part of $f$ will give us a value of 1, and on the circle of radius 1/2, the real part of $f$ will give us a value of 1/4. We can then compose with the function $g(z) = 40z - 30$ to get 10 and −10. Thus our complete function is the real part of

$$z \mapsto 40[\mathrm{Log}(z)]^2 - 30.$$

# References

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[2] Dan Kalman. General equation of an ellipse. Last visited on 2018-11-30.

[3] Eric W. Weisstein. Inverse hyperbolic tangent. From MathWorld—A Wolfram Web Resource. Last visited on 2018-11-29.