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Poking the Monster: Worthwhile or Whimsical?

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193,883 + 1 = 193,884. This seems like quite a trivial and elementary calculation, and technically, it is. However, this simple equation is an observation that helped establish the significance of the Monster group to mathematics as a whole. Although the Monster is an aspect of group theory, which is a vast area of mathematics, the majority of the information and research about the Monster has taken place in the last forty years, and thus, it potentially still has many mysteries to discover and add to the incomplete information obtained so far. Although it is a fascinating group by itself, one of the most interesting aspects of the Monster group is its unusual and subtle correlation to areas of mathematics and science that seem completely unrelated. It is a simple enough group to grasp even if its applications are nuanced, difficult, and at times deceiving, but it is practically the culmination of the entire field of group theory and, thus, requires prior knowledge of certain aspects of group theory before discussion of this fascinating object can begin.

The discussion of these topics must begin with normal subgroups and finite simple groups. First, "[a] subgroup H of the group G is called a normal subgroup if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$" (Beachy & Blair 157). In addition, it is clear that for any group G, both the group G itself and the set containing the identity of G are always normal, so for clarity, normal subgroups other than these will be referred to as nontrivial normal subgroups. Normal subgroups have a variety of applications, but for now, their primary use is in defining finite simple groups.

What are finite simple groups? On a technical level, they are simply finite groups—that is, groups with a countable number of elements—that contain no nontrivial normal subgroups. However, this does not at first glance convey their significance. As mathematician Terry Gannon describes them, "finite simple groups are to finite groups what the primes are to integers—they are their elementary building blocks" (*Moonshine Beyond* 4). Thus, finite simple groups are at the foundation of all other finite groups, and this has prompted substantial and detailed study in them.

In studying these groups, classifying them quickly became a task of utmost importance. The man who started this process, Wilhelm Killing, was working on a slightly different project, but as a prerequisite, his task involved "finding all the 'simple' ones and placing them in families," leading to "a 'periodic table' of…groups" (Ronan 65-66). However, these groups were infinite, but this work provided a foundation for classifying the simple finite groups into different families. Eventually, this classification of families was completed and proven to be complete, but "although there were no other families, there *were* some unexpected exceptions" (Ronan 96). Unsurprisingly, since each family simply describes the type of groups within it, the eighteen families in the periodic table are each infinite, but in addition to these, there are 26 of the above mentioned unexpected exceptions, which are known as "sporadic" groups (Gannon, *Moonshine Beyond*, 4). These 26 do not fit anywhere into the table of groups, and this is where the Monster group comes into play.

First, what is the Monster group? Simply put, it is the largest of the exceptions to the classification of simple finite groups, and this is an understatement. First, the Monster is intimately connected with many of the exceptional finite simple groups. As the name implies, the Baby Monster is connected with the Monster and actually was an integral part to discovering and constructing it (Ronan 178). However, it is not alone because "[t]he Monster involves all but six of the other exceptions" (247). This by itself is interesting, but the sheer size of the group relative to the others is also remarkable. At one point in the process of finding sporadic groups, the largest one was "of size 1,255,205,709,190,661,721,292,800. This means more than a million million million million symmetries" (161), which seems large to put it lightly. Next came what is known as the Baby Monster group, "which makes…other monsters seem small by comparison" (175), but the Monster group itself has $2^{46}\cdot3^{20}\cdot5^{9}\cdot7^{6}\cdot11^{2}\cdot13^{3}\cdot17\cdot19\cdot23\cdot29\cdot31\cdot41\cdot47\cdot59\cdot71$ elements, which is over 194 quintillion times larger than the Baby Monster (184). This is vast, and although these numbers may seem random at first, both the Monster's construction and its order are actually quite significant.

From this information, it is clear that the Monster itself is interesting in a way by virtue of being one of only 26 exceptions to classification and its role and properties in the context of the those exceptions. However, these qualities are not what make the Monster interesting from a mathematical point of view. Although its applications are advanced and difficult to see at first, this group has fascinating though subtle connections to other fields of mathematics and physics, and the fact that these connections are so unusual and tough could be seen as a benefit of studying the group more. As will shortly be discussed, the correlations between the Monster group and areas outside of group theory at first seem coincidental and inconsequential, but mathematicians have discovered true applications and mathematical motivation behind some of them. Because of this, it seems possible that since the Monster is still unexplained in many ways, it could yield yet more applications as of yet unknown to the world.

Before moving on to examples of these applications, it is vital to mention the Leech Lattice. It is less important to know how it came about than to know what its function was. A lattice is essentially a grid or structure of points that are all an equal distance away from each other, but this particular lattice packs spheres together in 24 dimensions (Ronan 148). However, what is remarkable about the Leech Lattice was that "[i]t is the tightest possible lattice packing in 24 dimensions…, [and] each 24-dimensional sphere touches 196,560 others" (Ronan 148). Because of the unique design and construction, this lattice had a vast number of symmetries and allowed mathematicians to directly construct twelve different sporadic groups, including some of the largest ones found at the time (155). Eventually, it would be instrumental in discovering and constructing the Monster group, but it would also eventually have interesting implications for the Monster's significance to the rest of the world.

Another foundational concept for the applications found in the Monster group is the idea of representation and character tables. Gannon states, "[a] *representation* of a group G is the assignment of a matrix $R(g)$ to each element g of G in such a way that the matrix product respects the group product, that is $R(g)\,R(h) = R(gh)$. The dimension of a representation is the size n of its $n \times n$ matrices $R(g)$" (*Moonshine Beyond* 4). Essentially, a representation is simply a way of looking at the group as matrices while still preserving the way the group operation behaves. Ronan describes a character table as "a square array of numbers that gives an immense amount of useful information" (135) about any group, and Gannon clarifies that these numbers "are the dimensions of the smallest irreducible representations" (*Moonshine Beyond* 4). Thus, one of the most helpful pieces of information a character table gives is the dimensions that the group representation can operate in, and the Monster's connection with number theory shows this result.

This connection takes place with modular groups and modular functions. Gannon describes modular functions as "functions living on complex curves" (*Moonshine Beyond* 2), and the originally observed function that the Monster connects to is a modular function called the *j*-function. The *j*-function has the form "$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$" (Conway and Norton, 309), and this is where the simple equation from the beginning of this paper comes in. Mathematician John McKay noticed that the first two elements of the character table for the Monster group, which again would be the two lowest dimensions in which the Monster can act according to the character table, are 1 and 196,883, which sum to the second coefficient of the *j*-function, and a pattern ensued (Ronan 192). This pattern amounted to the revelation "that there is a vertex operator algebra…; its automorphism group is the Monster and its graded dimension is the *j*-function (-744)" (Gannon, "Monstrous Moonshine" 1). As a side note, the "(-744)" is simply the means by which the constant "744" is removed from the *j*-function form. Essentially, this discovery meant that the connection between these two objects, the *j*-function and the Monster, could take a concrete form and actually exist, and this was new and significant as the two objects came from different areas of mathematics. Since this correlation between modular forms and the Monster group was first noticed, "the explanation and generalisation [sic] of this unlikely connection" is known as Moonshine (Gannon, "Monstrous Moonshine" 5).

In addition, the modular group also formed a connection with the Monster. According to Conway and Norton, a mathematician "noticed that the primes p dividing [the order of the Monster]…are just those for which the function field determined by the normalizer of $\Gamma_o(p)$ in PSL_2 (R) has genus zero" (308). This last part refers to patterns in the modular group, which "operates on the hyperbolic plane" (Ronan 198): each prime number corresponds to a group that is based on the modular group, and a genus of zero means that the surface generated by a given group is spherical (198). Again, this is unexpected, mysterious, and powerful. The long series of powers of prime numbers used to describe the size of the Monster mentioned previously is the same string of primes that corresponds to the groups involved with the modular group which all produce a sphere in functions from number theory. Granted, this is because the prime numbers "yield…mini-*j*-functions" (199), but while there are close ties between the mini-*j*-functions and the *j*-function, they were obtained by looking at different objects: one by looking at modular groups and the other at modular functions.

However, both of these instances deal with number theory, but the Monster has other interesting applications as well. The first step of understanding these applications is to discuss vertex operator algebras. Based in physics, these structures are designed to imitate many of the principles in string theory (Ronan 218). Gannon describes this structure as "an infinite-dimensional vector space V with infinitely many heavily constrained bilinear products" ("Monstrous Moonshine" 13). This type of space is a prerequisite to finally proving the connection between the *j*-function and the Monster group in the first place: that if "$V = \oplus_{n\in\mathbf{Z}} V_n$ is the infinite dimensional graded representation of the monster simple group…[t]hen…V satisfies the main conjecture in Conway and Norton's paper" (Borcherds 406). This main conjecture involves the above discussion of modular groups with genus zero, so the integral ideas of Moonshine, though they had much evidence behind them, did not have headway in being proven until concepts like vertex operator algebras came onto the scene (Ronan 218). It was at this point that applications into physics came into play.

One of the primary ways that the Monster applies to physics is as a result of principles concerning the Leech Lattice and an unusual geometry (Ronan 224-225). First, the idea of space-time came about during this time with Minkowski geometry of the form "$x^2 + y^2 + z^2 - t^2$," which meant that if "the square of the 'time-distance'" is zero, two events separated by space can be connected by a ray of light and are equivalent events (221-222). "When we expand to higher dimensional space-time…such a space is called Lorentzian" (223), and this type of space will occur again in this work soon. Here is where the Leech Lattice becomes more significant. Since the Leech Lattice is 24-dimensional, it is interesting to note Ronan's discussion of this number of dimensions; "here is a remarkable fact: $1^2 + 2^2 + 3^2 + 4^2 + \ldots + 21^2 + 22^2 + 23^2 + 24^2 = 70^2$[….] Twenty-four is the only whole number larger than 1 for which it happens. The sum of the first *n* squares is never a perfect square otherwise" (224). This is the foundation for an interesting quality of the Leech Lattice and, by extension, the Monster group, and this quality will be examined shortly.

With this in mind, the Monster has interesting connections to string theory. For one, Richard Borcherds, the one responsible for proving the validity of the main conjecture concerning Monstrous Moonshine and modular groups, used methods quite similar to physics in order to prove this conjecture (Ronan 225). Eventually, he demonstrated that "a string moving in space-time…'turns out to be non-zero only if space-time is 26-dimensional'" (225), and this matches nicely with some string theory. In Bosonic string theory, there exists "a critical dimension ($d = 26$) in which the bosonic string can consistently propagate" (Blumenhagen 35), and more significantly, "space-time Lorentz invariance of the quantized bosonic string in Minkowski state requires…$d = 26$" (47). What is important to note is that "Borcherds used the crystalline structure of the 26-dimensional Lorentzian Lattice…in creating his…algebra" (Ronan 224). This is significant because "[a] light ray in Lorentzian space – meaning a path on which the 'time-distance' is always zero – yields a 'perpendicular' Euclidean space of two dimensions lower" (223). Thus, any light ray in 26-dimensional Lorentzian space will produce a 24-dimensional Euclidean space, and now, taking the fact about 24 from above, a point with coordinates zero to 24 and with $t = 70$ "lies on a light ray through the origin…[and t]his light ray yields the Leech Lattice" (224). The fact that 24—the one integer which works with the previously mentioned series—is also a significant number for the Leech Lattice, string theory, Minkowski geometry, and the Monster group is incredible, and this fact truly seems to indicate the substantial importance of the Monster group.

However, how many of these connections truly matter or add to the Monster group's value as an area of study? According to Blumenhagen, "superstrings" actually have ten dimensions as a vital number of dimensions as opposed to the 26 dimensions significant in Bosonic string theory (243). So do these connections and findings mean much? Some are hopeful. Physicist Franklin Potter states, "if a 4^{th} quark family exists, the physical rules of the Universe follow directly from mathematical properties dictated by the…Monster group via the Monster's *j*-invariant function" (47). At the end of this work, he asserts his belief that not only is this implication true but that a 4^{th} quark family does exist, and now it remains only to find it (54). On the other end of the spectrum, Gannon shows slightly more skepticism. "Although there have been some attempts to directly interpret Monstrous Moonshine in the context of physics, we still have no evidence Nature concurs" (433). In many of these findings on the Monster's properties, it simply seems that not enough information is available to make a confident assertion as to whether or not many of these properties are useful. As Ronan reflects, "Perhaps it is only a coincidence that there are so many coincidences, but we do not know" (228). Perhaps the number 26 which occurs so many times truly is just a coincidence rather than an indicator of deeper meaning. All that can be done is simply to continue to study and wait for the results as Potter optimistically recommends (54).

Ultimately, the Monster group is still surrounded by a shroud of mystery. Some believe that it will help unlock understanding of the world itself, and some believe that it is most likely less significant than it seems. Ronan makes an insightful point when he states that "[s]trange connections…were not the reason mathematicians discovered the Monster, but a consequence" (228), and this is indicative of how mathematics can be approached. As of right now, the Monster group primarily provides potential, not definite answers, but that is not a reason to abandon it. In fact, that may be all the more reason to pursue it. While much of the world is still driven by task completion for the sake of progress, which includes mathematics so often, the Monster provides a unique opportunity: to pursue mathematics for the love of mathematics rather than simply as a means to an end. This is not to say that the Monster will yield no new results—based on the incredibly interesting implications of the group so far, it seems likely that it will continue to produce new and possibly useful results—but who knows? What is important about the Monster group? It is a mystery that will not likely be solved any time soon. That in and of itself merits more study.

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