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Tangent Lines

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In [1] Leibniz published the first treatment of the subject of calculus. An English translation can be found in [2]; he says that

> to find a *tangent* is to draw a right line, which joins two points of the curve having an infinitely small difference, or the side of an infinite angled polygon produced, which is equivalent to the *curve* for us.

Today, according to [3],

> a straight line is said to be a tangent line of a curve $y = f(x)$ at a point $x = c$ on the curve if the line passes through the point $(c, f(c))$ on the curve and has slope $f'(c)$ where f' is the derivative of f.

At first glance, it would appear that this second definition is simply a more precise version of the first; indeed, the cited Wikipedia article states this sentiment explicitly. In this paper we examine cases where Wikipedia's definition is more strict than Leibniz's original one, and present two attempts at formulating a more general, but still precise, definition.

1 Non-differentiability

It is well known that differentiability implies continuity:

$$f'(c) \text{ exists} \quad \implies \quad \lim_{x \to c}[f(x) - f(c)] = f'(c) \lim_{x \to c}(x - c) = 0$$

This makes intuitive sense; in order to be able to draw a tangent line to a curve at a point, that curve needs to actually be what we think of as a curve.

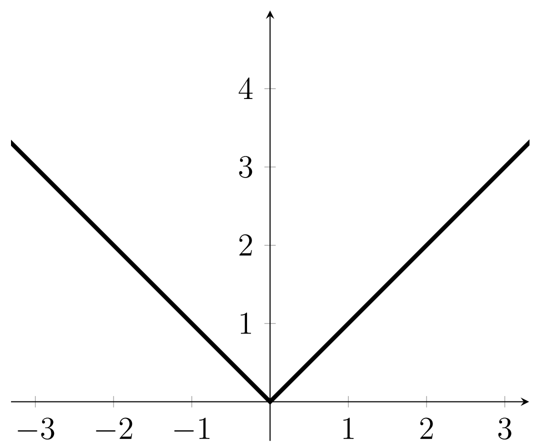

Figure 1: The absolute value function.

Implicit in Leibniz's definition is the assumption that we can find points with an infinitely (arbitrarily) small difference, which is exactly what continuity gives us. Thus we will assume continuity in our exploration here. One can imagine, especially given the next section, situations in which continuity could be deemed too strong, but those are beyond our scope.

1.1 Semi-differentiability

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$, as shown in Figure 1. We know that $f'(x) = \operatorname{sgn}(x)$ for $x \neq 0$, but what about that corner point at $x = 0$? It is tempting to extend this derivative to be zero here, so as to agree with the sign function everywhere; this is what the symmetric derivative does:

$$f_s(0) = \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h} = \lim_{h\to 0} \frac{|h| - |-h|}{2h} = 0$$

However, if we do this then we stray from our original motivation—namely, tangent lines. A more conservative approach is to just say that the left and right derivatives of f are

$$\partial_- f(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = -1 \quad \text{and} \quad \partial_+ f(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = 1;$$

since these both exist, f is called semi-differentiable at zero.

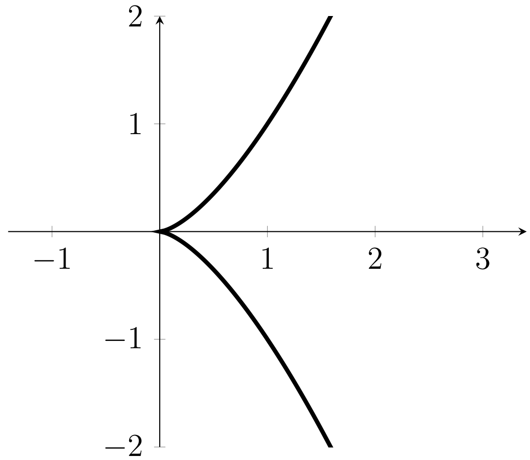

Figure 2: A semicubical parabola, exhibiting a cusp at the origin.

How does this relate to Leibniz's definition? At least in the translation above, an indefinite article is used to describe the term "tangent", which suggests that it need not be unique. Perhaps we should say that both $y = x$ $y = -x$ are tangent lines to $y = |x|$ at $x = 0$. But then again, this doesn't seem to line up quite so well with his imagery of an infinite-angled polygon.

1.2 Cusps

Consider the parametric curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\mathbf{r}(t) = (t^2, t^3)$, as shown in Figure 2. Then the derivative $\mathbf{r}'(t) = (2t, 3t^2)$ is defined everywhere, and for $t \neq 0$ it gives us a nonzero vector that tells us the tangent line to the curve. Note that $\mathbf{r}'(0) = (0, 0)$, so we could simply say that the tangent line to this curve is undefined at the origin. In fact, other parametrizations such as $t \mapsto (\sqrt[3]{t^2}, t)$ are actually non-differentiable at the origin.

But $\mathbf{r}$ does have a tangent line at the origin. If we instead think of this curve using the multivalued function $f(x) = \pm x^{3/2}$ for $x \geq 0$, then $f'(x) = \pm\frac{3}{2}\sqrt{x}$, so $f'(0) = 0$. This could obviously be made more rigorous using actual functions, but it gets the point across: the curve has a horizontal tangent line at the origin, and a unique one at that. It doesn't play with the conventional concept of derivatives, but if we go back to Leibniz's definition, we see that lines joining the origin to nearby points do indeed approach the x-axis as the difference approaches zero.

2 Tangentiability

Here we will attempt to make rigorous the notion of being able to define a unique tangent line to a curve at a point. The reader is invited to look for the interesting ways in which this intuitive interpretation breaks down when we relax the constraint of continuity.

2.1 Vector calculus

Let $\mathbf{r} : S \subseteq \mathbb{R} \to \mathbb{R}^2$ and $t \in S$. If the solution set to the equation

$$\lim_{h \to 0} \mathbf{r}_\perp \cdot \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=t} [\mathbf{r}_\perp \cdot \mathbf{r}(\tau)] = 0$$

is $\{c\mathbf{r}_\perp \mid c \in \mathbb{R}\}$ for some nonzero $\mathbf{r}_\perp = (x, y) \in \mathbb{R}^2$, then we say that $\mathbf{r}$ is tangentiable at t with orthogonal vector $\mathbf{r}_\perp$ and tangent vector $\mathbf{r}_\parallel = (-y, x)$.

Note first that this is an extension of differentiability whenever the derivative is nonzero. Specifically, if $\mathbf{r}'(t) \neq 0$ exists then $\mathbf{r}$ is tangentiable at t with tangent vector $\mathbf{r}_\parallel = \mathbf{r}'(t)$. To see this, consider the equation

$$\lim_{h \to 0} \mathbf{r}_\perp \cdot \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}_\perp \cdot \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}_\perp \cdot \mathbf{r}'(t) = 0.$$

Since $\mathbf{r}'(t) \neq 0$, we see that choosing $\mathbf{r}_\parallel = \mathbf{r}'(t)$ gives a nonzero $\mathbf{r}_\perp$ such that the solution set to this equation is $\{c\mathbf{r}_\perp \mid c \in \mathbb{R}\}$.

The converse is false, though. Consider the curve $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ given by $\mathbf{r}(t) = (|t|, |t|^{3/2} \operatorname{sgn}(t))$, which we can see to be simply another parametrization of the semicubical parabola from earlier. Then $\mathbf{r}'(t) = (\operatorname{sgn}(t), \frac{3}{2}\sqrt{|t|})$ for $t \neq 0$. If $\mathbf{r}'(0)$ existed then it would cause a jump discontinuity, which is impossible for a derivative; thus $\mathbf{r}$ is not differentiable at zero. But for any $\mathbf{r}_\perp = (x, y)$,

$$\lim_{h \to 0} \mathbf{r}_\perp \cdot \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h \to 0} \left[x \operatorname{sgn}(h) + y\sqrt{|h|}\right] = x \lim_{h \to 0} \operatorname{sgn}(h)$$

is zero iff $x = 0$, so the solution set to our definitional equation is $0 \times \mathbb{R}$. In other words, by our above definition of tangentiability, this $\mathbf{r}$ has a horizontal tangent line at $\mathbf{r}(0) = (0, 0)$.

Notice the specificity of the parametrization that we used for this example. If we use the $t \mapsto (t^2, t^3)$ parametrization the derivative at zero is defined and is itself zero, so the solution set is $\mathbb{R}^2$; in other words, our definition is not satisfied because the tangent line is not unique. On the other hand, if we use $t \mapsto (\sqrt[3]{t^2}, t)$ then one coordinate of the derivative tends to infinity while the other stays constant at 1, so the solution set is $\{(0,0)\}$. Thus, while our definition works correctly for some parametrizations, it fails for others, which is quite undesirable; we would like for our characterization of tangent lines to be as geometric as possible, and maximally agnostic of what parametrization we choose.

2.2 Complex numbers

We will now redefine tangentiability as follows. First, consider the function $\psi : \mathbb{C}^\times \to \mathbb{T}$ given by $\psi(z) = (z/|z|)^2 = z/\bar{z}$, where

$$\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \quad \text{and} \quad \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$$

are the multiplicative group of the complex numbers and the circle group, respectively. Clearly ψ is a homomorphism; let us examine its kernel. If $\psi(z) = 1$ then $(z/|z|)^2 = 1$, so $z/|z| = \pm 1$. This is equivalent to saying that z is an (obviously nonzero) point on the "horizontal" real line $\mathbb{R}$. What does this mean? Recall that multiplication in the complex numbers can be thought of geometrically as scaling and rotation. Thus given that the kernel is a horizontal line through the origin, we might guess that the cosets in $\mathbb{C}^\times/\ker\psi$ are all horizontal lines through the origin.

Let's make this rigorous. One direction is simple: for $z \in \mathbb{C}^\times$ and $c \in \mathbb{R}^\times$,

$$\psi(cz) = \frac{cz}{\overline{cz}} = \frac{cz}{c\bar{z}} = \frac{z}{\bar{z}} = \psi(z).$$

Conversely, write $z, w \in \mathbb{C}^\times$ in polar form as $z = ae^{i\theta}$ and $w = be^{i\phi}$ for $a, b \in \mathbb{R}^\times$ and $\theta, \phi \in \mathbb{R}$. If we assume that $\psi(z) = \psi(w)$ then

$$\psi(ae^{i\theta}) = \psi(be^{i\phi}) \quad \Longrightarrow \quad \frac{ae^{i\theta}}{ae^{-i\theta}} = \frac{be^{i\phi}}{be^{-i\phi}} \quad \Longrightarrow \quad e^{i(2\theta)} = e^{i(2\phi)},$$

which means that $2\phi - 2\theta = n(2\pi)$ for some $n \in \mathbb{Z}$, so $\phi = \theta + n\pi$. Then

$$(e^{i\pi})^n = (-1)^n = \pm 1 \quad \Longrightarrow \quad w = be^{i\phi} = be^{i(\theta+n\pi)} = \frac{b}{a}ae^{i\theta}(e^{i\pi})^n = \pm\frac{b}{a}z.$$

Since $\pm b/a \in \mathbb{R}^\times$, this shows that ψ partitions $\mathbb{C}^\times$ into lines passing through the origin (sans the origin itself, of course).

Now for $f : \mathbb{R} \to \mathbb{C}$ and $t \in \mathbb{R}$, let $S_t = \{s \in \mathbb{R} \mid f(s) \neq f(t)\}$. If t is a limit point of S_t, then we define the tangent of f at t to be

$$f_{\|}(t) = \lim_{s \to t} \psi(f|_{S_t}(s) - f(t))$$

if it exists. Since $\mathbb{T}$ is closed, we know $m_f(t) \in \mathbb{T}$. If we consider $\mathbb{C} \cong \mathbb{R}^2$ as vector spaces over $\mathbb{R}$, then the existence of this limit means that we can draw a tangent line to the graph of f at t.

We will show that this is an extension of our previous definition involving the dot product, and then show that it works even in the cases where our previous definition failed. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be tangentiable with nonzero orthogonal vector $\mathbf{r}_\perp = (x, y) \in \mathbb{R}^2$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(t) = (1, i) \cdot \mathbf{r}(t)$, and for any $t \in \mathbb{R}$, let $S_t \subseteq \mathbb{R}$ be as above. Assume that t is not a limit point of S_t, so there is some neighborhood $V \subseteq \mathbb{R}$ containing t such that $f(V) = \{f(t)\}$; this implies that $\mathbf{r}'(t) = 0$, so the solution set of orthogonal vectors must be $\mathbb{R}^2$, contradicting tangentiability. Thus t must be a limit point of S_t.

It remains to show that $f_{\|}(t) = \psi((1, i) \cdot \mathbf{r}_{\|})$. Besides the algebraic details, the key point here is that

$$\mathbf{r}'(t) = \lim_{s \to t} \frac{\mathbf{r}(s) - \mathbf{r}(t)}{s - t} \neq 0 \qquad \Longrightarrow \qquad \lim_{\substack{s \to t \\ s \in S_t}} \frac{s - t}{\|\mathbf{r}(s) - \mathbf{r}(t)\|} \text{ is bounded}$$

which allows us to show that

$$\begin{aligned}
f_{\|}(t) - \psi((1, i) \cdot \mathbf{r}_{\|}) &= \lim_{s \to t} \psi(f|_{S_t}(s) - f(t)) - \psi(-y + xi) \\
&= \lim_{\substack{s \to t \\ s \in S_t}} \frac{f(s) - f(t)}{[\Re f(s) - \Re f(t)] + [\Im f(s) - \Im f(t)]i} - \frac{-y + xi}{-y - xi} \\
&= \lim_{\substack{s \to t \\ s \in S_t}} \frac{2x[\Re f(s) - \Re f(t)] + 2y[\Im f(s) - \Im f(t)]}{(x - yi)([\Re f(s) - \Re f(t)] - [\Im f(s) - \Im f(t)]i)} \\
&= \frac{2}{x - yi} \lim_{\substack{s \to t \\ s \in S_t}} \frac{(x, y) \cdot (\Re[f(s) - f(t)], \Im[f(s) - f(t)])}{|f(s) - f(t)|} \\
&= \frac{2}{x - yi} \lim_{\substack{s \to t \\ s \in S_t}} \frac{\mathbf{r}_\perp \cdot [\mathbf{r}(s) - \mathbf{r}(t)]}{\|\mathbf{r}(s) - \mathbf{r}(t)\|} \\
&= \frac{2}{x - yi} \lim_{\substack{s \to t \\ s \in S_t}} \mathbf{r}_\perp \cdot \frac{\mathbf{r}(s) - \mathbf{r}(t)}{s - t} \lim_{\substack{s \to t \\ s \in S_t}} \frac{s - t}{\|\mathbf{r}(s) - \mathbf{r}(t)\|} \\
&= 0.
\end{aligned}$$

Knowing now that this second definition is more general than our earlier one, let's go back to the other two parametrizations of the semicubical parabola. If we define $f : \mathbb{R} \to \mathbb{C}$ by $f(t) = t^2 + t^3 i$ then $f(t) \neq f(0)$ for all $t \neq 0$, so

$$f_{\parallel}(0) = \lim_{s\to 0} \psi(f(s) - f(0)) = \lim_{s\to 0} \frac{s^2 + s^3 i}{s^2 - s^3 i} = \lim_{s\to 0} \frac{1 + si}{1 - si} = 1 = \psi(1);$$

in other words, the tangent line to f at zero is the unique line through the origin containing 1, that is, the real line. Or if we define it by $f(t) = \sqrt[3]{t^2} + ti$ then, again, $f(t) \neq 0$ for all $t \neq 0$, so

$$f_{\parallel}(0) = \lim_{s\to 0} \psi(f(s) - f(0)) = \lim_{s\to 0} \frac{\sqrt[3]{s^2} + si}{\sqrt[3]{s^2} - si} = \lim_{s\to 0} \frac{1 + \sqrt[3]{s}i}{1 - \sqrt[3]{s}i} = 1 = \psi(1).$$

This, needless to say, is pretty rad.

References

[1] Gottfried Leibniz. Nova methodus pro maximis et minimis. *Acta Eruditorum*, X, October 1684.

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