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Steady State Probabilities in Relation to Eigenvalues

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Steady State Probabilities in Relation to Eigenvalues

Eigenvalues and eigenvectors have numerous useful applications in mathematics and physics. One specific application is the use of eigenvalues to calculate steady-state probabilities and mean return times of ergodic chains. In "Operations Research, an Introduction," Hamdy Taha uses the method of determining these probabilities from the constraints: $\vec{\pi} = \vec{\pi}\mathrm{P}$ and $\sum_j \pi_j$ where P is a stochastic matrix in which the row probabilities sum to 1. The author then solves the system of equations with these constraints and uses the solution to determine the steady state probabilities and mean return times (Taha 528). These steady state probabilities can be very useful in determining expected values and obtaining other results. It is clear from the initial constraint $\vec{\pi} = \vec{\pi}\mathrm{P}$ that this represents an eigenvalue problem as this can be restated as:

$$\vec{0} = \vec{\pi}\mathrm{P} - \vec{\pi}$$

$$\vec{\pi}(\mathrm{P} - \mathrm{I}) = \vec{0}$$

Thus, this appears to be an eigenvalue problem in which 1 is an eigenvalue. However, this is notably different than the equations presented in "Linear Alegebra, 3rd Edition" by Fraleigh. There are differences between Taha's technique of computing the steady state vector and Fraleigh's technique of taking $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$ to find the eigenvalue from the characteristic equation and then compute the eigenvectors (Fraleigh 290-91). Upon closer examination of these steady state problems, many questions arise in relation to the consistent result of an eigenvalue equal to 1, the potential ability or inability to reproduce Taha's results from Fraleigh's method of using determinants, and the existence of eigenvalues other than 1.

The initial difference between Taha's method and the techniques learned in linear algebra is that the eigenvectors computed in linear algebra appear on the right side of the unknown vector which are to be solved. Therefore, to ensure the same results, one must transpose the stochastic matrix P so that the columns sum to 1 instead of the rows:

$$\vec{\pi} = \vec{\pi}\mathrm{P}$$

$$\vec{\pi} = \mathrm{P}^{\mathrm{T}}\vec{\pi}$$

$$(\mathrm{P}^{\mathrm{T}} - \mathrm{I})\vec{\pi} = \vec{0}$$

which moves $\vec{\pi}$ to the right side. This right eigenvector follows Fraleigh's method.

After manipulating this to the form of a right eigenvalue problem, one is able to solve for the eigenvector $\vec{\pi}$, given that $\lambda = 1$. But to truly make this an eigenvalue problem, one must consider $(\mathrm{P}^{\mathrm{T}} - \lambda\mathrm{I})\vec{\pi} = \vec{0}$ and compute λ. Taha's method assumes that $\lambda = 1$, but why is this always the case? When dealing with a stochastic matrix, the column entries all sum to 1. Subtracting 1 from the diagonal, as is the case of $\lambda = 1$, it is obvious that the columns sum to zero. Thus, since the matrix $(\mathrm{P}^{\mathrm{T}} - \mathrm{I})$ has columns that sum to zero, the sum of the row vectors is 0 and the rows are linearly independent, proving that there is some nontrivial vector $\vec{\pi}$ that satisfies the equation $(\mathrm{P}^{\mathrm{T}} - \mathrm{I})\vec{\pi} = \vec{0}$ and thus 1 is always an eigenvector of this stochastic matrix P^{T}.

The assumption is lack of knowledge as to what λ is for $(\mathrm{P}^{\mathrm{T}} - \lambda\mathrm{I})\vec{\pi} = \vec{0}$ and therefore it follows to compute it using Fraleigh's method for a stochastic matrix P^{T} whose column components sum to 1. Looking specifically at example 17.4-1 (Taha, 578-579), one finds:

$$P = \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.4 & 0.55 \end{bmatrix} \text{ such that } P^T = \begin{bmatrix} 0.3 & 0.1 & 0.05 \\ 0.6 & 0.6 & 0.4 \\ 0.1 & 0.3 & 0.55 \end{bmatrix}$$

$$P^T - \lambda I = \begin{bmatrix} 0.3-\lambda & 0.1 & 0.05 \\ 0.6 & 0.6-\lambda & 0.4 \\ 0.1 & 0.3 & 0.55-\lambda \end{bmatrix}$$

Taking the determinant of $(P^T - \lambda I)$ and setting it equal to zero, the result is

$$\det(P^T - \lambda I) = (0.3-\lambda) * [(0.6-\lambda) * (0.55-\lambda) - (0.4) * (0.3)] - (0.1) * [(0.6) * (0.55-\lambda) - (0.4) * (0.1)] + (0.05) * [(0.6) * (0.3) - (0.6-\lambda) * (0.1)] = 0.$$

Using algebra to simply and solve for λ such that $\det(P^T - \lambda I) = 0$ produces:

$$-\lambda^3 + 1.45\lambda^2 - 0.49\lambda + 0.04 = 0$$

Using Mathematica to compute the roots,

```
In[6]:= Roots[-x^3 + 1.45 x^2 - 0.49 x + 0.04 == 0, x]

Out[6]= x == 0.121922 || x == 0.328078 || x == 1
```

one discovers the eigenvalues of 0.121922, 0.328078, and 1.

Considering the eigenvalue of 1 and computing the corresponding eigenvector, one must first consider the matrix:

$$P^T - 1I = \begin{bmatrix} -0.7 & 0.1 & 0.05 \\ 0.6 & -0.4 & 0.4 \\ 0.1 & 0.3 & -0.45 \end{bmatrix}.$$

The solution to the system is

$$\begin{bmatrix} -0.7 & 0.1 & 0.05 \\ 0.6 & -0.4 & 0.4 \\ 0.1 & 0.3 & -0.45 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Simplifying this matrix in Mathematica produces:

```
In[8]:= RowReduce[{{-0.7, 0.1, 0.05}, {0.6, -0.4, 0.4}, {0.1, 0.3, -0.45}}]

Out[8]= {{1, 0., -0.272727}, {0, 1, -1.40909}, {0, 0, 0}}.
```

Thus, there is a free variable for z which can define other variables as they relate to z. Choosing z=s, one has x=0.272727*s and y=1.40909*s and the eigenvector is

$$s\begin{bmatrix}0.272727\\1.40909\\1\end{bmatrix}.$$

This is the eigenvector for the eigenvalue of 1. However, it is obvious that this does not match the steady state probabilities obtained by Taha's methods. To obtain the same results, one must choose an s that causes the eigenvector to sum to 1 when dealing with probabilities. Thus, s is to be the inverse of the sum of all of the entries:

$$s = \frac{1}{(0.272727+1.40909+1)} = 0.37288152.$$

The new eigenvector is

$$s\begin{bmatrix}0.272727\\1.40909\\1\end{bmatrix} = 0.37288152\begin{bmatrix}0.272727\\1.40909\\1\end{bmatrix} = \begin{bmatrix}0.101694858\\0.525423621\\0.37288152\end{bmatrix},$$

which is equivalent to Taha's solution in example 17.2-1. Therefore, one is able to obtain the same results with linear algebra methods. Taha's method is simply an application of eigenvalues and eigenvectors. Adding the constraint $\sum_j \pi_j$ has the same impact as choosing an s such that the eigenvector entries sum to 1.

Clearly the eigenvalue is 1 which produces the same results as Taha. However, this is not the only eigenvalue for the stochastic matrix. In the previous example, there were three eigenvectors. Most notable about the other two eigenvectors is that some of the entries have opposite signs, making it impossible for them to represent some probability. The eigenvector $\vec{v}$ to the eigenvalue 1 is called the stable equilibrium distribution of the stochastic matrix A and is also called Perron-Frobenius eigenvector. It can be proven that for a stochastic matrix A, it will always have an eigenvalue of 1 and that all other eigenvalues will be less than 1. It does not seem as though these other eigenvalues carry any real significance as the eigenvalue of 1 does.

From this application, it is evident how helpful eigenvectors can be in solving problems. Taha used his knowledge and experience with eigenvalue problems to derive a shortcut to compute these steady state probabilities that consistently works every time. Significantly these stochastic matrices always have an eigenvalue of 1, and some other eigenvalues are less than 1.

Bibliography

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Taha, Hamdy A. Operations Research: An Introduction (9th ed). Upper Saddle River, NJ: Prentice-Hall, Inc., 2006.