A Brief Study of Some Aspects of Babylonian Mathematics

Tom Zara

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Sandra V. Rumore, Ph.D.
Chairman of Thesis

____
Glyn Wooldridge, Ph.D.
Committee Member

____
Randall Dunn, Ed.D.
Committee Member

____
James Nutter, D.A.
Honors Director

____
Date

Abstract

Beginning over 4000 years ago, the Babylonians were discovering how to use mathematics to perform functions of daily life and to evolve as a dominant civilization. Since the beginning of the 1800s, about half a million Babylonian tablets have been discovered, fewer than five hundred of which are mathematical in nature. Scholars translated these texts by the end of the 19th century. It is from these tablets that we gain an appreciation for the Babylonians' apparent understanding of mathematics and the manner in which they used some key mathematical concepts. Through this thesis, the author will provide background information about the Babylonians and then explain the manner in which the Babylonians used a number system, the square root of 2, "Pythagorean" mathematics, and equations.

A Brief Study of Some Aspects of Babylonian Mathematics

Background

Between the years of 3500 B.C. and 539 B.C., various Mesopotamian civilizations inhabited this "land between the rivers" (Dellapena, 1996, p. 213) of the Euphrates and the Tigris (see *Figure 1* below for a map of this region). Around 3500 B.C., the Sumerians established the first city-states; one of the best city-states was called Ur. After the Sumerians came the Akkadians, who inhabited the area of the surrounding desert. The Akkadians were conquered in about 1900 B.C. by the First Babylonian Empire. Just over 1000 years later, in 885 B.C., the Assyrians took over the land from the Akkadians and maintained control of the land for nearly 300 years until, in 612 B.C., the Chaldeans conquered the Assyrians and began the Second Babylonian Empire. Unlike the First Babylonian Empire, the Chaldeans' reign was short-lived, a mere 73 years, until the Persians invaded the land in 539 B.C. (Teresi, 2002). For a timeline of these events, see *Figure 2*.

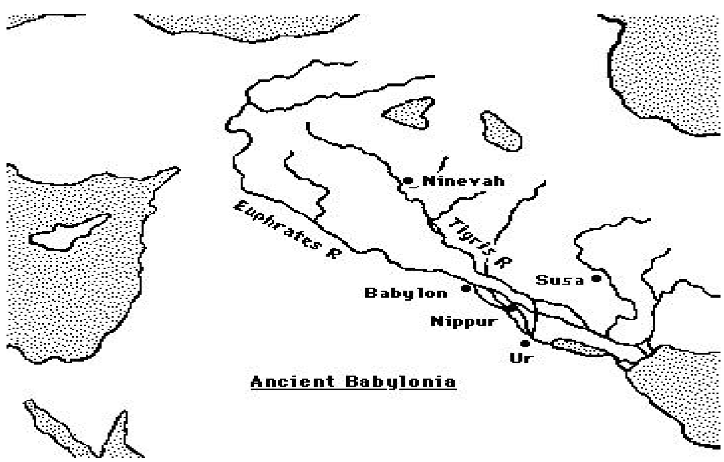

Figure 1. A map of Ancient Babylonia.[1]

[1] From "Pythagoras's Theorem in Babylonian Mathematics," by J.J. O'Connor and E.F. Robertson, 2000, MacTutor History of Mathematics, ¶ 2.

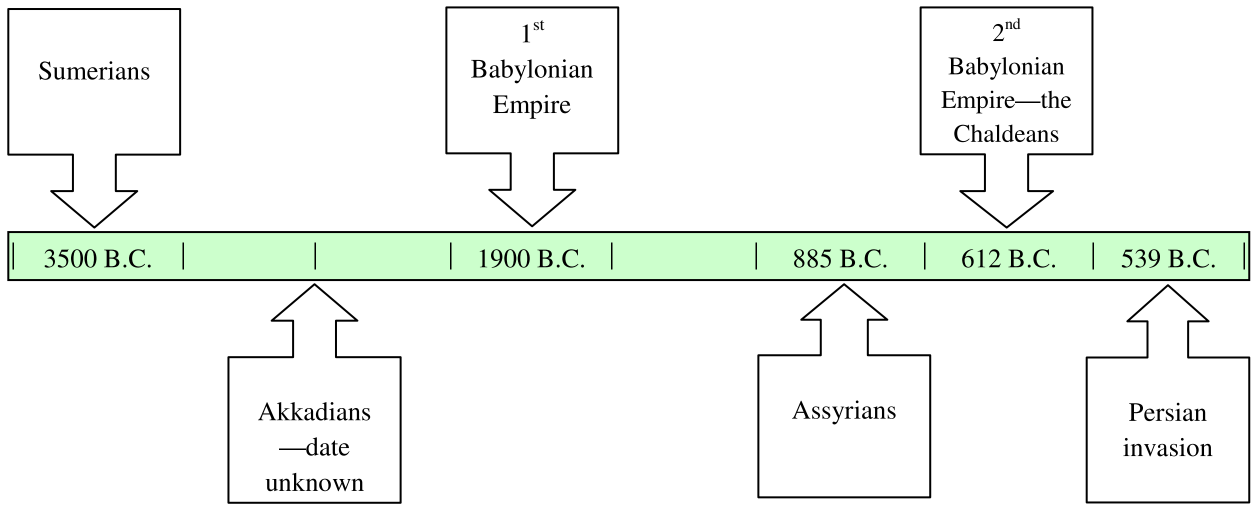

Figure 2. Chronology of the Mesopotamian civilizations in Ancient Babylonia.

Mathematical Contributions in Mesopotamia

In this area of Ancient Babylonia, mathematical contributions were made by these Mesopotamian civilizations. When discussing the mathematical contributions made in Mesopotamia, the entire period from 3500 B.C. to 539 B.C. is referred to as the Babylonian era; however, when the contributions are determined to have been made during the earliest period of the Mesopotamian civilizations, the term “Sumerian” is used (Teresi, 2002).

The information we have regarding Babylonian mathematics comes from clay tablets. Although approximately half a million of these tablets have been discovered since the beginning of the 1800s, fewer than five hundred are mathematical in nature (Teresi, 2002). The majority of these five hundred tablets are dated between the years 1800 and 1600 B.C. It was not until the end of the 19th century, however, that numerous Sumerian and Babylonian measurement texts were translated. Nevertheless, by the late 1920s the study of Babylonian mathematics was well-established and scholars attained a thorough understanding of the methods Babylonian mathematicians implemented for solving problems (Høyrup 2002).

Formation of Babylonian Clay Tablets

The script that was used on the clay tablets is called cuneiform script and the texts were written in the Babylonian language, which is a dialect from the Akkadians that is Semitic in nature and is closely related to the classical Arabic and Hebrew languages. The secret for the great preservation of these Babylonian tablets lies in the manner in which the information was written. The scripts were written on moist clay tablets using a stylus, which is a blunt reed. The clay was then baked, either by the sun or in an actual oven. The impressions that remained were wedge-shaped, which is the reason for the name of these scripts—"cuneiform," which literally translates "wedge shaped." Among the various Mesopotamian civilizations, the Sumerians were the first to establish a system of writing using this cuneiform method, primarily for bureaucratic purposes. Despite the benefit of the great preservation of these scripts due to this method of inscription, many tablets contain several errors since the scribes had to write on the moist clay very quickly before the clay dried (Teresi, 2002).

It is from these well-preserved tablets that we gain our understanding of the number system the Babylonians had in place, their dealings with "Pythagorean" mathematics and equations, possible ways they determined the value of the square root of 2, and some other mathematical topics.

To begin our brief review on some of the Babylonian mathematics, we are going to look at the Babylonian number system.

The Number System

In most parts of the world today, a decimal place value system that uses the Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is used. The position of these numerals affects the value of the number. For example, in the numbers 6, 60, and 600 the numeral 6 is in three different places—in the first number, the six is in the units place, with the value of 6×10^0; in the second number, the six is in the tens place, with the value of 6×10^1; and in the third number, the six is in the hundreds place, with the value of 6×10^2. However, the Babylonians developed a number system that was sexagesimal in nature, which means that instead of having a base of ten (decimal), it had a base of 60 (Hodgkin, 2005). The modern-day methods for measuring time, geographic coordinates, and angles follow such a sexagesimal system. For example, the angle measure of 4°1'15" is equivalent to $4 + (1/60) + (15/60^2)$, the sum of which is $4.0208\overline{33}°$. However, the Babylonians did not have a pure 60-base system, since they did not use 60 individual digits; rather, they counted by both 10s and 60s. Therefore, in reality, the Babylonians' notation system may be considered both a decimal and sexagesimal system (Teresi, 2002).

When the Sumerians established this system, it was incomplete in the sense that they used positional notation only in base 60. As *Figure 3* shows, the Sumerians only had the following symbols:

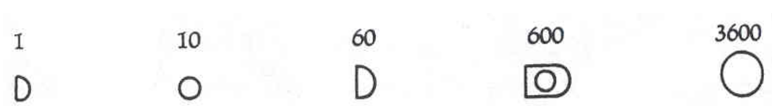

Figure 3. The symbols the Sumerians used prior to 2000 B.C.[2]

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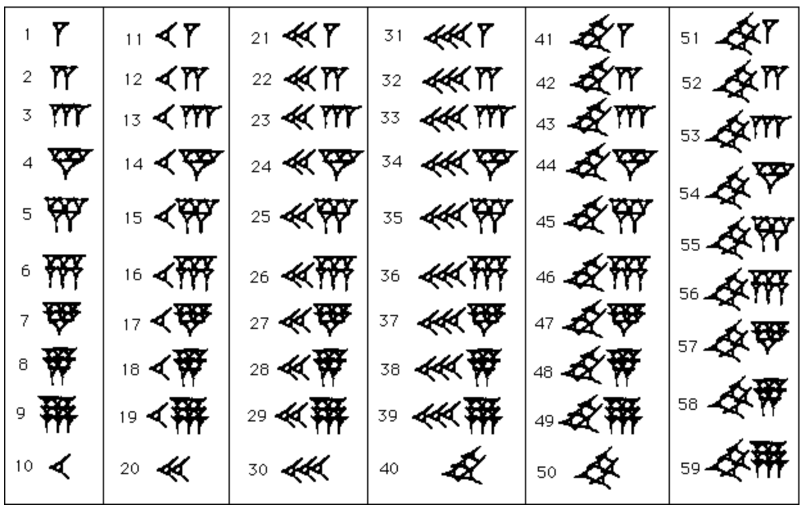
[2] From *Lost discoveries*, by D. Teresi, 2002, p.48.

However, in about 2000 B.C., a simpler number system was devised by the Babylonians. In this system, only two symbols were used: a pin shape that represented a value of one, and a wing shape that represented a value of 10 (Teresi, 2002). *Table 1* shows how numbers under 60 were written.

Table 1. *The numbers from 1 through 59 written in the cuneiform script.*[3]



From about 2500 B.C. on, the Babylonians' number system drastically improved when they realized that the pin- and wing-shaped symbols could represent various values based on their position in relation to each other. In this place-value system, the manner in which values were represented was by placing the signs side by side. Also, the Babylonian number system is read from left to right (Teresi, 2002). So the number 95, for example, would be written as follows:

[3] From "Babylonian Numerals," by J.J. O'Connor & E.F. Robertson, 2000.

This first pin shape represents a value of 60, the three wings are each worth 10 (3 X 10 = 30), and the final five pins are each worth one (5 X 1 = 5), which results in a total of 95 (Teresi, 2002).

Although this variation of the number system utilized the positioning of symbols to alter their values, this method too had its limitations. For example, instead of using a symbol like zero as a placeholder to represent an "empty column" between two numbers, the Babylonians' "placeholder" was simply leaving extra space between their number symbols. To add to the complexity of this system, the value of a symbol differed based on its size; so a symbol written slightly smaller than whatever was considered "standard" at the time would have a different value than a larger variation. Consequently, a reader's misinterpretation of the writer's size of symbols or spacing between symbols could easily lead to mistakes regarding the symbols' value and even whether the symbol represented a fraction or a whole number (Teresi, 2002).

In order to better understand the value of these symbols, editors usually transliterate[4] the value and add commas or semicolons to signify and distinguish between whole numbers and decimals, respectively. This practice began with the pioneer scholar Otto Neugebauer[5] in the 1930s (Teresi, 2002). From the transliteration in which commas are used, the transliterated value can be turned into a decimal value by multiplying the

[4] According to the *Oxford English Dictionary* (1989), to transliterate is "[t]o replace (letters or characters of one language) by those of another used to represent the same sounds."

[5] Neugebauer (1899-1990) was an Austrian-American historian of science and mathematician in the 19th century.

number on the far right by 60^0, the number immediately to its left by 60^1, the number immediately to the left of the previous number by 60^2, etc., and then taking the sum of these values. For example, the decimal value equivalent of the transliteration '1, 15' is $15 \times 60^0 + 1 \times 60^1 = 75$. Similarly, '44, 26, 40' has a value of $40 \times 60^0 + 26 \times 60^1 + 44 \times 60^2 = 40 + 1560 + 158{,}400 = 160{,}000$ (Hodgkin, 2005).

While commas are used in the transliteration of whole numbers, semicolons are used in the transliteration of decimal fractions[6]. In the transliterated value of the Babylonian number, the semicolon signifies a "decimal point," even though the Babylonians had not yet established a symbol for this concept. The transliteration of a number in which semicolons are used can be turned into a decimal value by dividing the first number to the right of the semicolon by 60^1, the number immediately to the right of the previous number by 60^2, the number immediately to the right of the previous number by 60^3, etc., and then taking the sum of these values. For example, '1; 20' is calculated as $1 + (20/60) = 4/3$; or 0; 30 would be equivalent to $0 + (30/60) = 0.5 = 1/2$. Another example would be '1; 24, 51, 10,' which is equivalent to $1 + (24/60^1) + (51/60^2) + (10/60^3)$. When these terms are added together, the sum is $1.41421\overline{296}$. This value will prove to be essential later on in this work in the author's explanation of a key Babylonian tablet (Hodgkin, 2005).

The transliterations of Babylonian symbols by editors have helped readers to better understand the values of the symbols written in cuneiform script. However, not all editors come up with the exact same transliterations. This is due to the way each editor

[6] According to *Mathematics Dictionary* (James, James, & Alchian, 1976), a decimal fraction is "a number that in decimal notation has no digits other than zeros to the left of the decimal point" (p. 98).

interprets the spacing between symbols—namely, whether or not there is the indication of a "zero"—along with the size of the symbols. For example, '4 12' may be transliterated in a variety of ways—as 4, 12 = 12 X 60^0 + 4 X 60^1= 252, as 4; 12 = 4 + (12/60) = (21/5), or as 4; 1, 2 = 4 + ($1/60^1$) + ($2/60^2$) = $4.017\overline{22}$. Similarly, since the Babylonians did not have a decimal point to separate the integer and fractional parts of a number nor a symbol for zero, the numbers 160, 7240, $2\,{}^2/_3$, and ${}^2/_{45}$ were all written in the exact same way (Teresi, 2002). *Table 2* below provides examples of the transliterations and the decimal value equivalents for some larger cuneiform numbers.

Table 2. *Transliterations and decimal values for some larger cuneiform numbers*.[7]

Cuneiform	Transliteration	Decimal value
	1,15	75
	1,40	100
	16,43	1003
	44,26,40	160000
	1,24,51,10	305470

Somewhere between the years of 700 and 300 B.C., the Babylonians made an improvement in their number system by implementing a symbol that would mean "nothing in this column" (Teresi, 2002, p. 50). This development was a step toward the modern usage of zero as a placeholder. However, in this particular model the Babylonians used a symbol of two little triangles arranged in a column to represent the placeholder between two other symbols. This new symbol helped eliminate some of the ambiguity that existed in their previous form of the number system. For example, the number 7,240 could now be written as follows:

7 From *A History of Mathematics: From Mesopotamia to Modernity*, by L.H. Hodgkin, 2005, p. 23.

Without the placeholder symbol, such a number could be calculated as 160—2 pin shapes, each of which have a value of 60 (2 X 60 = 120) plus 4 wing shapes, each of which have a value of 10 (4 X 10 = 40) for a total of 160 (120 + 40 = 160). However, since the placeholder symbol is in the 60s column, the pin shapes become worth 60^2 each instead of just 60^1. The wings still have a value of 10 each, which implies that the value is (2×60^2) + (4 X 10), which results in a sum of 7,240 (Teresi, 2002).

Since the placeholder symbol was never placed at the end of numbers, but rather was used only in the middle of numbers, it appears that the placeholder symbol never evolved into an actual symbol for zero. However, the Babylonians' use of this placeholder symbol has still proven to be helpful for editors in translating symbols (Teresi, 2002).

In addition to the evolution of the Babylonians' number system, another topic of interest is the Babylonians' apparent understanding of the number $\sqrt{2}$.

The Square Root of 2

One perplexing tablet that has been discovered is the Yale tablet YBC[8] 7289. Although the exact time this tablet was written is unknown, it is generally dated between 1800 and 1650 B.C. On this tablet, there is evidence that the Babylonians may have had an understanding of irrational numbers—particularly, that of $\sqrt{2}$ (O'Connor & Robertson, 2000).

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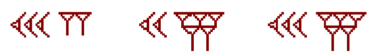
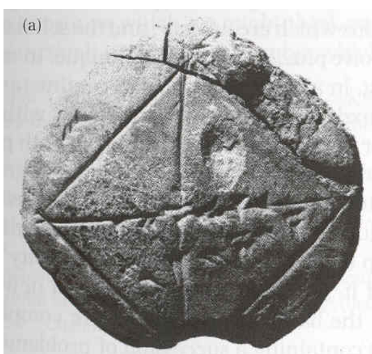
[8] YBC stands for Yale Babylonian Collection, which is an independent branch of the Yale University Library located in New Haven, Connecticut in the United States. The YBC consists of over 45,000 items, which makes it the largest collection in the Western Hemisphere for Near Eastern writing.

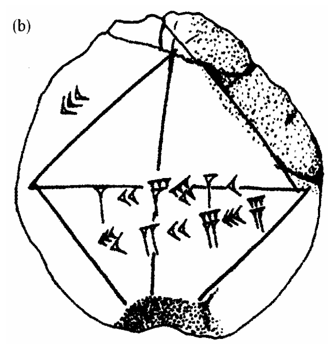
Engraved in the tablet is the figure of a square, with one side marked with the number 30 (see *Figure 4* below). In addition, the diagonal has two sexagesimal numbers marked—one of which is

and the other of which is



Regarding the former of these two numbers[9], scholars agree on transliterating it as 1; 24, 51, 10, which is approximately $\sqrt{2}$ (1; 24, 51, 10 is equal to $1 + {}^{24}/_{60^1} + {}^{51}/_{60^2} + {}^{10}/_{60^3}$, the sum of which is $1.41421\overline{296}$), accurate to five decimal places (Hodgkin, 2005).





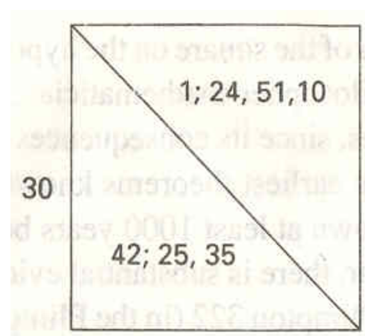


Figure 4(a). YBC 7289 tablet.[10] *Figure 4(b).* Drawing. *Figure 4(c).* Dimensions.[11]

However, sources vary regarding the value of the second of these two diagonals. This discrepancy is due to the manner in which the numbers are transliterated. For example, when transliterated as 0; 42, 25, 35, the value is ${}^{42}/_{60^1} + {}^{25}/_{60^2} + {}^{35}/_{60^3}$

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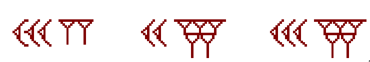
[9] Both of these sets of symbols were copy and pasted from "Babylonian numbers" (Edkins 2006).
[10] Figures 4(a),(b) both from *A History of Mathematics: From Mesopotamia to Modernity*, by L.H. Hodgkin, 2005, p. 25.
[11] From *The History of Mathematics Brief Version*, by V.J. Katz, 2004, p. 17.

≈ 0.7071064815, which is ${}^{\sqrt{2}}/_{2}$ accurate to six decimal places[12]. However, other sources transliterate the number as 42; 25, 35 (as is shown in *Figure 4(c)*), which is equal to 42 + ${}^{25}/_{60^1} + {}^{35}/_{60^2} = 42.4263\overline{88}$[13]. This is the equivalent of $30\sqrt{2}$, accurate to three decimal places. Such a calculation implies that this value was determined by multiplying the length of the side (30) by the length of the diagonal ($\sqrt{2}$).

It seems more logical to this author that the latter transliteration of ‘42 25 35’ to 42; 25, 35 is the correct one. The reasoning behind such a conclusion is based on the fact that the object appears to be that of a square, with one of the sides being labeled with a value of 30. Based on the geometrical definition of a square,[14] each of the remaining sides must also have a value of 30. With the diagonal being drawn in such a way as to equally divide the square into two right triangles, the two remaining triangles are each of type 45°-45°-90°. This implies that the three sides for each of these two triangles are related to each other by the proportion $x{:}x{:}x\sqrt{2}$, with x representing the measure of the two equal legs and $x\sqrt{2}$ representing the measure of the hypotenuse. By definition, since the two legs have already been determined to have a measure of 30, the length of the hypotenuse must be $30\sqrt{2}$. A potential explanation as to why the value of 1; 24, 51, 10 (namely, $\sqrt{2}$) was inscribed in a position so close to 42; 25, 35 (i.e., $30\sqrt{2}$) is that $\sqrt{2}$ may have served as an indication of how the value of $30\sqrt{2}$ was derived.

A possible reason for the transliteration of  to 0; 42,

[12] This is the way that Hodgkin (2005, p. 25) and Høyrup (2002, p. 262) present the value of this diagonal.
[13] This is the way that O’Connor and Robertson (2000) present the value of this diagonal; Katz (2004, p. 16) is a proponent of this view as well.
[14] According to *Mathematics Dictionary* (James et al., 1976), a square is “a quadrilateral with equal sides and equal angles” (p. 362).

25, 35, which is about 0.7071064815 ($\sqrt{2}/_2$ accurate to six decimal places), may be based on an alternate transliteration of 30—the value of the side of the square inscribed on the tablet. Some scholars transliterate 30 as 0; 30 = $^{30}/_{60}$ = $^{1}/_{2}$[15]. Even so, such a transliteration still does not line up with right triangle trigonometry because this transliteration would indicate that the sides of the triangle are related by the proportion $^{1}/_{2}:^{\sqrt{2}}/_{2}:\sqrt{2}$, which does not satisfy the Pythagorean Theorem[16]. Therefore, this alternate transliteration seems incorrect.

Regardless of the manner in which these numbers are transliterated, one can conclude that the sexagesimal numbers

 and 

are of importance, as they appear again in the work of Islamic mathematicians over 3000 years after this Babylonian work. While it appears that Babylonian mathematicians were able to use irrational numbers like $\sqrt{2}$, scholars have not come to an agreement regarding how the Babylonians derived these values (Hodgkin, 2005).

Theories for the Derivation of $\sqrt{2}$.

In "Pythagoras's Theorem in Babylonian Mathematics," Robertson (2000) proposes a method for how the Babylonians arrived at their approximation of $\sqrt{2}$. He suggests that since the Babylonians used tables of squares and seem to have based multiplication around squares, they may have made two guesses, say *a* and *b*, where *a* is

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[15] Hodgkin (2005, p. 25) and Høyrup (2002, p. 260) both present this transliteration, although Hodgkin only states that it is an alternate value for 30, whereas Høyrup says that it is "probably 30'." (p. 260)

[16] *Mathematics Dictionary* (James et al., 1976) states that, according to the Pythagorean theorem, "[t]he sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse" (p. 312).

a low number and b is a high number. After taking the average of these two numbers and squaring that average, which is $[(a + b) / 2]^2$, if the result were greater than 2, then b could be replaced by this better bound. However, if the value were less than 2, then a could be replaced by $(a + b)/2$. The algorithm would then continue to be carried out. Such a method takes several steps to get a fair approximation of $\sqrt{2}$. For example, it takes 19 steps to get to the sexagesimal value of 1; 24, 51, 10 when $a = 1$ and $b = 2$, as is evident by *Table 3* below:

Table 3. *Nineteen iterations of an algorithm for computing an approximation of* $\sqrt{2}$.[17]

step	decimal	sexagesimal
1	1.500000000	1;29,59,59
2	1.250000000	1;14,59,59
3	1.375000000	1;22,29,59
4	1.437500000	1;26,14,59
5	1.406250000	1;24,22,29
6	1.421875000	1;25,18,44
7	1.414062500	1;24,50,37
8	1.417968750	1;25,4,41
9	1.416015625	1;24,57,39
10	1.415039063	1;24,54,8
11	1.414550781	1;24,52,22
12	1.414306641	1;24,51,30
13	1.414184570	1;24,51;3
14	1.414245605	1;24,51;17
15	1.414215088	1;24,51;10

[17] From "Pythagoras's Theorem in Babylonian Mathematics," by J.J. O'Connor & E.F. Robertson, 2000.

16	1.414199829	1;24,51;7
17	1.414207458	1;24,51;8
18	1.414211273	1;24,51;9
19	1.414213181	1;24,51;10

Although this method may seem very tedious, since the Babylonians were excellent at making computations, it should not necessarily be ruled out (O'Connor & Robertson, 2000).

Differing from Robertson's suggested method for how the Babylonians came to such an accurate approximation of $\sqrt{2}$, many authors theorize that the Babylonians used a method equivalent to a method Heron used.[18] The conjecture is that the Babylonians began with some guess for the value of $\sqrt{2}$, which we will call x. Then they calculated e, the error: $e = x^2 - 2$. Then $(x - e/2x)^2$ can be expanded to the equivalent expression $x^2 - e + (e/2x)^2$. By adding the number two to both sides of the equation for e, the error, and replacing x^2 in the previous expression with $e + 2$, we find that the expression can be written as $2 + (e/2x)^2$, which produces a better approximation of $\sqrt{2}$, since if e has a small value then $(e/2x)^2$ will be even smaller. Equation (1) shows the progression of this expression:

$$(x - e/2x)^2 = x^2 - e + (e/2x)^2 = 2 + (e/2x)^2 \tag{1}$$

By continuing this process, the approximation for $\sqrt{2}$ gets more and more accurate. In fact, if one starts with the value of $x = 1$, only two steps of the algorithm are necessary to

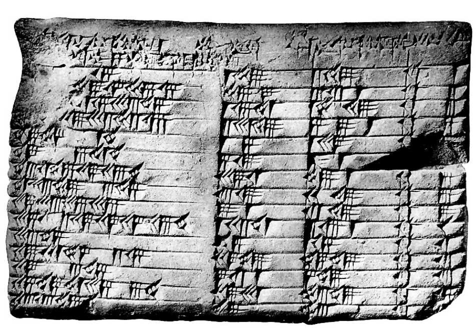
[18] Heron of Alexandria (or Hero of Alexandria) was a geometer during the first century who invented various machines and whose best known work in mathematics is the formula for finding the area of a triangle based on the lengths of its sides.

get a value that is equivalent to the approximation 1; 24, 51, 10. The fact that the Babylonians used quadratic equations, which we will look at more thoroughly later on, makes this a plausible method for finding the approximation of $\sqrt{2}$. This algorithm, however, is not evident in any other cases; so although it may be a plausible method, it is not necessarily likely (O'Connor & Robertson, 2000).

If, in fact, the previous method for finding the approximation for $\sqrt{2}$ is accurate, then the Babylonians appear to have been familiar with Pythagorean mathematics. Another well-known tablet provides support for this theory.

"Pythagorean" Mathematics

Of all the tablets that reveal Babylonian mathematics, the most famous is arguably one that has been named "Plimpton 322"—a name given to it because it possesses the number 322 in G.A. Plimpton's Collection at Columbia University. In terms of the tablet's size, it is small enough to fit in the palm of one's hand (Rudman, 2007). This tablet is believed to have been written around 1800-1700 B.C. in Larsa, Iraq (present-day Tell as-Senkereh in southern Iraq) and it was first cataloged for the Columbia University Library in 1943 (Katz, 2004). As is evident in *Figure 5* below, the upper left corner of this tablet is damaged and there is a large chunk missing from around the middle of the right side of the tablet (O'Connor & Robertson, 2000).



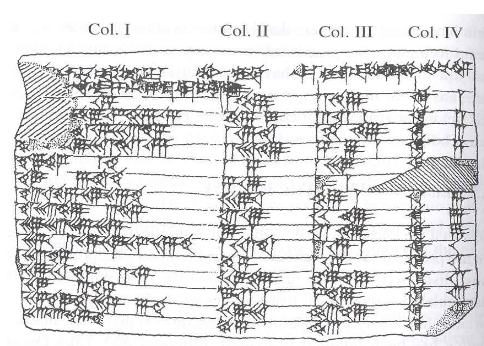


Figure 5(a). The Plimpton 322 tablet. [19] *Figure 5(b).* A drawing of Plimpton 322. [20]

This tablet has four columns, which we will refer to as Column I…Column IV, and 15 rows that contain numbers in the cuneiform script. Column IV is the easiest to understand, since it simply contains the row number, from 1 through 15. Column I, however, is often considered an enigma due to the missing information caused by the damage in the left corner of the tablet. In *Mathematical Cuneiform Texts*, Neugebauer and Sachs make note of the fact that in every row, when the square of each number x from Column II (see *Table 4* below) is subtracted from the square of each number d from Column III, the result is a perfect square, say y. In the original tablet, the heading for the values that we denote x from Column II can be translated as "square-side of the short side" and the heading for the values that we denote d from Column III can be translated as "square-side of the diagonal" (Katz, 2004, p. 18). This can be translated into the following equation:

$$d^2 - x^2 = y^2 \tag{2}$$

Consequently, many scholars argue that the numbers on this particular tablet serve as a listing of Pythagorean triples[21] (O'Connor & Robertson, 2000). These triples are listed in their translated decimal form in *Table 4* below.

Table 4. *Pythagorean triples from the Plimpton 322 tablet.* [22]

[19] From "Pythagoras's Theorem in Babylonian Mathematics," by J.J. O'Connor & E.F. Robertson, 2000.
[20] From *How Mathematics Happened: The First 50,000 Years*, by P.S. Rudman, 2007, p. 216.
[21] Pythagorean triples are whole numbers that satisfy the equation $a^2 + b^2 = c^2$—where, in a right triangle, a and b represent the lengths of two sides that are perpendicular to each other and where c represents the length of the hypotenuse—which is referred to as the Pythagorean theorem.

$(d/y)^2$	x	d	#	y
1.9834028	119	169	1	120
1.9491586	3367	4825	2	3456
1.9188021	4601	6649	3	4800
1.8862479	12,709	18,541	4	13,500
1.8150077	65	97	5	72
1.7851929	319	481	6	360
1.7199837	2291	3541	7	2700
1.6845877	799	1249	8	960
1.6426694	481	769	9	600
1.5861226	4961	8161	10	6480
1.5625	45	75	11	60
1.4894168	1679	2929	12	2400
1.4500174	161	289	13	240
1.4302388	1771	3229	14	2700
1.3871605	28	53	15	45

Although there are only four columns in the actual Plimpton 322 tablet, *Table 4* makes use of an additional column—which we will refer to as Column V—that contains values equal to the square root of $d^2 - x^2$, namely the middle value for each of the Pythagorean triples. Although the values of Column I cannot be known for certain because of the damage in this area of the tablet, most scholars agree that each of these values is the quantity of the value from Column III (which is labeled d) over the value from Column II (which is labeled x), all of which is squared, as is depicted in *Table 4* above. In *A Contextual History of Mathematics*, Calinger (1999) explains that many historians have considered Column I to have some kind of connection to the secant function (O'Connor & Robertson, 2000).

While *Table 4* seems to make it evident that Plimpton 322 is, in fact, a listing of Pythagorean triples, the reader should be aware that not all the decimal values in this table are accurate translations of the symbols written in cuneiform script in the original tablet. In order to accept the theory of the tablet being a listing of Pythagorean triples, one would have to conclude that the author(s) of the tablet made four inscription errors,

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[22] From *The History of Mathematics Brief Version*, by V.J. Katz, 2003, p. 18.

two in each column. The values in *Table 4* are based on what are considered to be the corrected values. For example, in row six of the original tablet the scribe gave *d* in Column III a value of 9, 1 which is equivalent to 1 X 60^0 + 9 X 60^1; this value is equal to 541. However, this appears to be an inscription error since the Pythagorean triple that would correspond with the value of 319 for *x* in row six would be 319(*x*), 360(*y*), 481(*d*). The value shown in *Table 4* for *d*, which is located in Column III, is produced from the transliteration of 8, 1 which is equivalent to 1 X 60^0 + 8 X 60^2; this value is equal to 481, which correctly satisfies the Pythagorean triple for row six. In addition to the inscription errors on this tablet, there does not appear to be a logical ordering of the rows, except that the numbers in Column I decrease with each successive row (O'Connor & Robertson, 2000).

An advocate of the theory that Plimpton 322 is a listing of Pythagorean triples, Erik Christopher Zeeman[23], made an interesting observation that may confirm that Plimpton 322 actually contains Pythagorean triples. Zeeman observed that if the Babylonians had used the formulas $h = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$ for producing Pythagorean triples, then there are 16 triples that satisfy the conditions: $30° \leq t \leq 45°$, $n \leq 60$, and $\tan^2 t = h^2/b^2$. Of these 16 triples that satisfy the previous conditions, 15 are listed in Plimpton 322 (O'Connor & Robertson, 2000).

While the theory of Pythagorean triples seems to be the most popular explanation of the Plimpton 322 tablet among scholars and historians, there are critics who oppose this view. For example, according to O'Connor and Robertson (2000), in "Babylonian Mathematics and Pythagorean Triads" Exarchakos states "... we prove that in this tablet

[23] Zeeman (1925 -) is a British mathematician who was born in 1925 in Japan. Zeeman is most well-known for his work in singularity theory and especially in geometric topology.

there is no evidence whatsoever that the Babylonians knew the Pythagorean theorem and the Pythagorean triads." Rather, Exarchakos believes that the tablet is connected to solutions for quadratic equations (O'Connor & Robertson, 2000).

For those who do accept that the tablet contains fifteen Pythagorean triples on it, this does not necessarily imply that the Babylonians had an understanding of the Pythagorean relationship in right triangles. In fact, Pythagorean triples may be viewed simply as a relationship among three geometric squares, as *Figure 6* below shows for the most well-known Pythagorean triple 3, 4, 5. Or, since the Babylonians seem to have been more algebraic than geometric in their approach to mathematics, they may have looked at Pythagorean triples as a relationship among squared integers. However, Neugebauer translated the heading to Column III as "diagonal," which implies that the Babylonians actually did view Pythagorean triples in relation to right triangles (Rudman 2007).

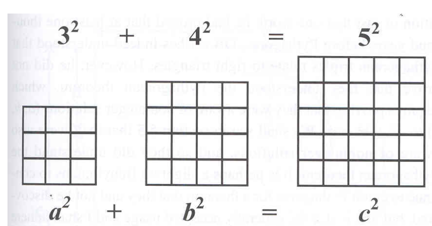

Figure 6. Geometric representation of Pythagorean triples. [24]

Another piece of evidence that points to the idea that the Babylonians understood the concept of Pythagorean triples and the Pythagorean theorem[25] is the translation of a Babylonian tablet that currently is being held in the British museum, which states:

[24] From *How Mathematics Happened: The First 50,000 Years*, by P.S. Rudman, 2007, p. 217.
[25] Note that Pythagoras, the person after whom the theorem is named, was not even born until about 1200 to 1500 years after the approximated date of the writing of this tablet.

4 *is the length and* 5 *the diagonal. What is the breadth* ?

Its size is not known.

4 *times* 4 *is* 16.

5 *times* 5 *is* 25.

You take 16 *from 25 and there remains* 9.

What times what shall I take in order to get 9 ?

3 *times* 3 *is* 9.

3 *is the breadth.* (O'Connor & Robertson, 2000, ¶ 1)

A modern translation of such a problem may be as follows:

In a right triangle, the length of one of the sides is 4, the hypotenuse has a measure of 5, and the remaining side—which we will denote x—is unknown. In order to find the measure of the unknown side, we set up an equation according to the Pythagorean Theorem—namely, the sum of the square of the length for each of the two legs is equal to the length of the hypotenuse squared. Based on the information given on this particular right triangle, the equation would be set up as follows:

$$4^2 + x^2 = 5^2 \tag{3}$$

Since 4^2 is 16 and 5^2 is 25, the equation may now be written in the following form:

$$16 + x^2 = 25 \tag{4}$$

In order to isolate the variable x, we first subtract 16 from both sides, which leaves us with $x^2 = 9$.

Now we take the square root of both sides of the equation and find that $x = 3$.

Even if the reader concludes that the Babylonians were aware of Pythagorean triples and the Pythagorean theorem, he may ask how the Babylonians were able to derive these numbers. The Babylonians may have gone about their derivation of these numbers in a manner comparable to the way Diophantus of Alexandria[26] did over 1500 years after Plimpton 322 was written. According to Diophantus' method, one begins with the definition of a triple—namely, the Pythagorean theorem with strictly integer terms: $a^2 + b^2 = c^2$. By rearranging the terms so that b^2 is isolated and then factoring on the right-hand side of the equation, we get the following:

$$b^2 = c^2 - a^2 = (c - a)(c + a) \tag{5}$$

A scribe may then have divided each of the terms in this factored form of the Pythagorean theorem by b^2 in order to arrive at the following reciprocal relation:

$(c/b + a/b)(c/b - a/b) = 1$, or equivalently, $(c/b + a/b) = 1/(c/b - a/b)$

Since a, b, c must be integers, $(c/b + a/b)$ and $(c/b - a/b)$ must be common fractions and thus can be expressed as : $(c/b + a/b) = p/q$ and $(c/b - a/b) = q/p$, where p and q are also integers. Now by simple addition and subtraction:

[26] Diophantus of Alexandria (between 200 and 214 to between 284 and 298) was a mathematician who wrote *Arithmetica*, a series of books that involve solving algebraic equations, and who is also often called the "Father of Algebra."

$$[(c/b + a/b) + (c/b - a/b)]/2 = c/b = (p/q + q/p)/2 \qquad (6)$$

$$[(c/b + a/b) - (c/b - a/b)]/2 = a/b = (p/q - q/p)/2 \qquad (7)$$

Using these results in the Pythagorean theorem, we obtain it in *triples* form:

$$\boldsymbol{a^2 + b^2 = c^2} \qquad (8)$$

$$\boldsymbol{(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2} \qquad (9)$$

All that remains is to choose integers for p and q to generate Pythagorean triples. (Rudman, 2007, p. 220)

As is evident in the manner in which Rudman proposes the Babylonians may have come about their derivation of the Pythagorean triples, Rudman assumes that the Babylonians had an understanding of second-degree equations. Such an assumption is logical since scholars uncovered Babylonian mathematical tablets with solutions for second-degree equations in the early twentieth century. Such findings showed that Babylonian mathematicians not only understood linear equations, which scholars had already known about for some time, but also equations of the quadratic type.

Equations

In the retrieved works of the Babylonians, we find the novel idea of representing an unknown quantity—for example, an unweighed stone (Hodgkin, 2005). Today, we use symbols (often letters) that represent some unspecified or unknown quantity. The Babylonians referred to their unknown quantity as *sidi*, for “side”—like the measure of the side of a square. Our modern equivalent for *sidi* would simply be “x.” Similarly, we use “x^2” where the Babylonians would use the word *mehr*, which means “square” (Teresi, 2002).

From these unknown quantities, the Babylonians would then proceed to find the values of the unknowns by setting up and solving a linear equation. However, there are limited examples available to us today of the Babylonians’ use of linear equations and they generally appear as a system of linear equations.

Like the Egyptians, the Babylonians primarily solved these equations through the method of false position. According to *Mathematics Dictionary* (James et al., 1976), the method of false position (also referred to as “regula falsi”) is

> A method for approximating the roots of an algebraic equation. Consists of making a fairly close estimate, say r, then substituting $(r + h)$ in the equation, dropping the terms in h of higher degree than the first (since they are relatively small), and solving the resulting linear equation for h. This process is then repeated, using the new approximation $(r + h)$ in place of r. *E.g.*, the equation $x^3 - 2x^2 - x + 1 = 0$ has a root near 2 (between 2 and 3). Hence we substitute $(2 + h)$ for x. This gives (when the terms in h^2 and h^3 have been dropped) the equation $3h - 1 = 0$; whence $h = 1/3$. The next estimate will then be $2 + 1/3$ or $7/3$. (p. 149)

An example of a scenario in which the Babylonians would use the method of false position for solving equations comes from the Old Babylonian text VAT 8389:

> One of two fields yields 2/3 *sila* per *sar*, the second yields 1/2 *sila* per *sar*. [*Sila* and *sar* are measures for capacity and area, respectively.] The yield of the first field was 500 *sila* more than that of the second; the areas of the two fields were together 1800 *sar*. How large is each field? (Katz, 2004, p. 21)

Such a problem can easily be translated into a system of two linear equations as follows:

$$(2/3)x - (1/2)y = 500 \tag{10}$$

$$x + y = 1800 \tag{11}$$

Using the method of false position, this Babylonian scribe assumed that both x and y were equal to 900, which satisfies equation (11). However, when these values are used in equation (10), the result is 150, which is 350 less than the desired result. At this point, Katz (2004) explains that

> [t]o adjust the answer, the scribe presumably realized that every unit increase in the value of x and consequent unit decrease in the value of y gave an increase in the "function" $(2/3)x - (1/2)y$ of $(2/3) + (1/2) = (7/6)$. He therefore needed only to solve the equation $(7/6)s = 350$ to get the necessary increase, $s = 300$. Adding 300 to 900 gave him 1200 for x, and subtracting 300 from 900 gave him 600 for y—the correct answers. (p. 21)[27]

A common algebraic equation during the First Babylonian period is as follows: "Multiply two-thirds of [your share of barley] by two-thirds [of mine] plus a hundred qa

[27] While the Babylonian scribe used false position to solve this system of linear equations, current mathematicians would probably use either the method of substitution or the method of elimination.

of barley to get my total share. What is [my] share?" (Teresi, 2002, p. 50). Such a problem is solved by the same technique we use for solving linear equations today (Teresi, 2002).

The tablet YBC 4652 contains another similar linear equation: "I found a stone, but did not weight it; after I added one-seventh and then one-eleventh [of the total], it weighed 1 *mina* [= 60 *gin*]. What was the original weight of the stone?" (Katz, 2004, p. 21). This particular problem can be translated into the modern equation

$$(x + x/7) + (1/11)(x + x/7) = 60 \tag{12}$$

Although the tablet does not contain the procedure the scribe followed for solving the problem, it does contain the correct answer of 48 $^1/_8$. However, based on what we know about the Babylonians' typical method for solving such linear equations, we can assume with confidence that the scribe probably used the method of false position (Katz, 2004).

Although by the later part of the nineteenth century there had been an established understanding among scholars of the Babylonians' use of a sexagesimal place value system and their ability to solve linear equations, it was not until the late 1920s at Otto E. Neugebauer's seminar in Göttingen that Babylonian solutions for second-degree equations were discovered. Prior to that point, second-degree equations were thought to have originated in India, which Indian mathematicians probably had borrowed from the Arabs. According to Høyrup (2002), in Neugebauer's journal, *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Neugebauer states:

> . . . we see that complex linear equation *systems* were drawn up and solved, and that Babylonians drew up systematically problems of *quadratic* character and certainly also knew to solve them – all of it with a computational technique that is wholly equivalent to ours. If this was the situation already in Old Babylonian times, hereafter even the later development will have to be looked at with different eyes. (p. 2)

Although the Babylonians' use of problems that can be translated into quadratic equations was not discovered until the twentieth century, the tablets containing problems of quadratic equations actually outnumber those containing linear problems (Katz, 2004). In fact, from as early as 2000 B.C., the Babylonians were able to solve systems of equations in the form

$$x + y = p \tag{13}$$

$$xy = q \tag{14}$$

If we solve the second equation for y (which produces the equation $y = q/x$), substitute this value of (q/x) for y in the first equation (which gives us $x + (q/x) = p$), and then multiply all the terms in this revised equation by x, we get the equivalent quadratic equation

$$x^2 + q = px \tag{15}$$

Then this system of equations was solved by the following method:

"(i.) Form $^{(x+y)}/_2$

(ii.) Form $[^{(x+y)}/_2]^2$

(iii.) Form $[^{(x+y)}/_2]^2 - xy$

(iv.) Form $\sqrt{[^{(x+y)}/_2]^2 - xy} = {}^{(x-y)}/_2$

(v.) Find x, y by inspection of the values in (i), (iv)" (Stillwell, 1989, p. 51)

in order to get the two roots—when both roots were positive, since the Babylonians did not use negative numbers—of the form

$$x, y = {}^{p}/_2 \pm \sqrt{({}^{p}/_2)^2 - q} \tag{16}$$

When the scribes solved these "quadratic-type problems"[28] they often utilized, as Katz (2004) calls it, "cut-and-paste" (p. 21) geometry that was already developed by surveyors. With this approach, the Babylonians were able to solve many typical problems such as determining the length and width of a rectangle when the semiperimeter and area are given (Katz, 2004). One such example from the tablet YBC 4663 contains a

[28] Hodgkin makes a distinction between the terms "quadratic-type problems" and the "quadratic equation," stating that the problems that the Babylonians worked with vary in nature and that "quadratic equations" in the modern sense did not truly come about until the Islamic period.

problem in which the information given can be translated into the following two equations:

$$x + y = 6\frac{1}{2} \tag{17}$$

$$xy = 7\frac{1}{2} \tag{18}$$

In order to solve this problem,

> The scribe first halves $6\frac{1}{2}$ to get $3\frac{1}{4}$. Next, he squares $3\frac{1}{4}$, getting $10\frac{9}{16}$. From this is subtracted $7\frac{1}{2}$, leaving $3\frac{1}{16}$, and then the square root is extracted to get $1\frac{3}{4}$. The length is thus $3\frac{1}{4} + 1\frac{3}{4} = 5$, while the width is given as $3\frac{1}{4} - 1\frac{3}{4} = 1\frac{1}{2}$. (Katz, 2004, p. 21)

This particular problem can be solved using a geometric procedure that the Babylonians may have utilized (see *Figure 7*[29] below).

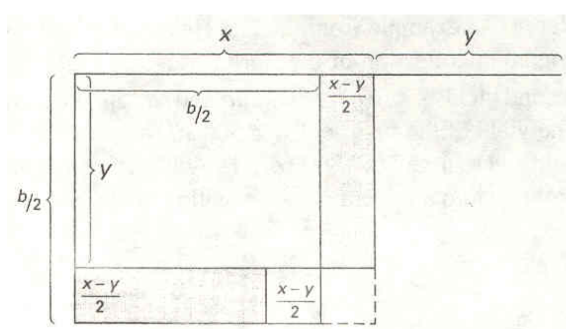

Figure 7. Geometric depiction for solving the system $x + y = b$, $xy = c$.[30]

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[29] In this particular depiction of the geometric figure, the sides of the figure are labeled based on the genetic system $x + y = b$, $xy = c$.

[29] From *The History of Mathematics Brief Version*, by V.J. Katz, 2003, p. 22.

The scribe began by halving the sum b and then constructing the square on it. Since $b/2 = x - (x - y)/2 = y + (x - y)/2$, the square on $b/2$ exceeds the original rectangle of area c by the square on $(x - y)/2$, that is,

$$\left[{}^{(x+y)}/_{2}\right]^2 = xy + \left[{}^{(x-y)}/_{2}\right]^2 \tag{19}$$

The figure then shows that adding the side of this square, namely, $\sqrt{(b/2)^2 - c}$, to $b/2$ gives the length x, and subtracting it from $b/2$ gives the width y. The algorithm is therefore expressible in the form

$x = (b/2) + \sqrt{(b/2)^2 - c} \quad y = (b/2) - \sqrt{(b/2)^2 - c}$. (Katz, 2004, p. 21-22) (20)

Another practical example of the Babylonians' use of quadratic-type problems is the following:

"I have added up seven times the side of my square and eleven times the area: 6; 15" (Hodgkin, 2005, p. 25).

What this translates to is a square in which seven times the unknown side x (which is $7x$) is added to eleven times the area (which is $11x^2$), which yields a result of 6; 15. This is equivalent to $6 + {}^{15}/_{60}$, which is equal to $6\,{}^{1}/_{4}$. Such a problem can be written as the basic quadratic equation $7x + 11x^2 = 6\,{}^{1}/_{4}$. An equation like this was then solved by a methodical process as follows:

> You write down 7 and 11. You multiply 6,15 by 11: 1,8,45. (Multiply the constant term by the coefficient of x^2.)
>
> You break off half of 7. You multiply 3,30 and 3,30. (Square half the x-coefficient.)
>
> You add 12,15 to 1,8,45. Result 1,21. (12,15 is the result of the squaring, so the 1,21 is what we would call $(b/2)^2 + ac$, if the equation is $ax^2 + bx = c$.)
>
> This is the square of 9. You subtract 3,30, which you multiplied, from 9. Result 5,30. (This is $-(b/2) + \sqrt{(b/2)^2 + ac}$; in the usual formula, we now have to divide this by $a = 11$, which we proceed to do.)
>
> The reciprocal of 11 cannot be found. By what must I multiply 11 to obtain 5,30? The side of the square is 30. ('Simple' division was multiplying by the reciprocal, for example, dividing by 4 is multiplying by 15, as we have seen. If there is no reciprocal, you have to work it out by intelligence or guesswork, as is being done here.) (Hodkin, 2005, p. 30)

While today we have the convenience of using a calculator for large calculations, such technology was not available to the Babylonians. Instead, they had tables with the values of squares, cubes, reciprocals, and square and cube roots. In addition, they had tables for the values of $x^3 + x^2$ for integer values from 1 to 20 as well as for the integers 30, 40, and 50 (Teresi, 2002). Considering the Babylonians' limitations in their method of computing, their ability to take linear and quadratic-type problems alike and come up with accurate solutions is an impressive accomplishment.

Conclusion

Through the approximately five hundred mathematical Babylonian tablets that have been discovered since the 1800s, we have been able to gain a better understanding and a deeper appreciation of the vast contributions that the Babylonian mathematicians made to the mathematics that exists today. Some of these areas in particular are their number system, their use of "Pythagorean" mathematics, their calculation of the square root of 2, and their use of equations. Present-day students and scholars alike are indebted to the Babylonian mathematicians and others who have laid a foundation of mathematics.

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