Teaching Conceptual Understanding of Mathematics via a Hands-On Approach

Jessica Korn

A Senior Thesis submitted in partial fulfillment
of the requirements for graduation
in the Honors Program
Liberty University
Fall 2014

Acceptance of Senior Honors Thesis

This Senior Honors Thesis is accepted in partial fulfillment of the requirements for graduation from the Honors Program of Liberty University.

____
Connie McDonald, Ph.D.
Thesis Chair

____
Esther Alcindor, M.Ed.
Committee Member

____
George Young, Ph.D.
Committee Member

____
James H. Nutter, D.A.
Honors Director

____
Date

Abstract

Given the current developments in the field of mathematics instruction in the United States, conceptual understanding and a hands-on approach in mathematics are two topics of importance. Conceptual understanding of mathematics is often lacking but characterizes the core of what mathematics actually is. Using a hands-on approach presents an effective way to teach conceptual understanding of mathematics. In order to argue this, a presentation of the underlying theories of mathematical understanding and pertinent approaches is given. Then follows an investigation of three studies pertinent to using a hands-on approach in teaching conceptual understanding of mathematics: one related to implementation (Gürbüz, Çatlioğl, Bìrgìn, & Erdem, 2010), one related to tools (Özgün-Koca & Edwards, 2011), and one related to evaluation (Bartell, Webel, Bowen, & Dyson, 2013). Furthermore, an example of a hands-on mathematical activity in geometry (Tipps, Johnson, & Kennedy, 2011) which could be implemented in a real-life classroom is presented. A discussion then follows, providing interpretation and implications, such as the connection to differentiation, to teaching conceptual understanding using a hands-on approach.

Keywords: mathematics instruction, conceptual understanding, hands-on approach, hands-on activities

Teaching Conceptual Understanding of Mathematics via a Hands-On Approach

The three Rs—reading, writing, and arithmetic—have been a part of the core of education in the United States for several centuries. Mathematics education is not only a part of a student's education in the United States but also is a part of a student's education in countries around the world. Thus, mathematics education is a topic of importance for teachers and students around the globe. However, mathematics is neither solely a set of algorithms nor solely rote memorization. Mathematics is a practical discipline reaching into a wide variety of fields. Thus, mathematics education should include appropriate emphasis on the teaching of conceptual understanding of mathematics. Using a hands-on approach presents an effective way to teach conceptual understanding of mathematics.

Presenting both an overview of the theory behind conceptual understanding and hands-on learning and then research on conceptual understanding and specifically hands-on learning provides the basis for discussion on the coupling of these two aspects of mathematics instruction. An over-arching rationale is that mathematics education is global. And thus, aiming to implement the teaching of conceptual understanding of mathematics applies not merely to educators in the United States but also to educators around the world. The rationale is that if students do not have as full of an understanding of mathematics as students could and should because educators inadequately teach conceptual understanding of mathematics, then the topic of teaching conceptual understanding of mathematics is of importance. Also, educators should have the knowledge and skills to teach conceptual understanding of mathematics. Furthermore, mathematical understanding itself is important given the following observation made by

J. Piaget: the seemingly contradictory situation in which students who do well academically struggle with mathematics despite the fact that “mathematics constitutes a direct extension of logic itself” (1970, p. 44). The current trends and concerns in mathematics education (specifically in relation to mathematics education in the United States)—warrant the given purpose and rationale.

Theory

Theories undergird and underlie every scholarly concept. The theories supporting the teaching of conceptual understanding of mathematics explore the nature of mathematics, understanding, and pedagogical methodologies. One assumption underlying these theories is that developing mathematical understanding is important: that a student solely having procedural knowledge of mathematics is not as valuable as if he or she had conceptual understanding of mathematics. Furthermore, a student having conceptual understanding of mathematics provides a more holistic education for him or her. These theories, with their assumptions support using a hands-on approach in teaching conceptual understanding.

Foundation of the Study of Mathematics

Investigating mathematics. Examining the teaching of mathematics first requires having a clear understanding of what mathematics itself is. “Mathematics, the science of patterns, is a way of looking at the world,” argued K. J. Devlin (1994, p. 6). Mathematics is a way of thinking,” according to J. C. Jones (2012, p. 4), or “a logical system of thinking,” according to W. A. Brownell (1947, p. 261). Thus mathematics is both an art and a science (Brownell, 1947; Devlin, 1994; Jones, 2012). One sees the patterns (Devlin, 1994) but then represents them in an organized way (Brownell, 1947;

Jones, 2012). In teaching mathematics in a classroom setting, the following provides a visual picture of mathematics, presented by Thurston (1990, p. 7):

> 'Mathematics isn't a palm tree, with a single long straight trunk covered with scratchy formulas. It's a banyan tree, with many interconnected trunks and branches—a banyan tree that has grown to the size of a forest, inviting us to climb and explore.' (as cited in Romberg & Kaput, 1999, p. 5)

Some might argue that in the present-day technological age—in the world of calculators, smartphones, and supercomputers—mathematics, or at least the study of mathematics, is obsolete. Although there are situations in which using a calculator is more convenient or in having a computer do the mathematical calculations is more efficient, mathematics and the study thereof is not obsolete. Merely entering numbers into a calculator does not indicate mathematical understanding. Mathematics is a valuable aspect of one's everyday life and evidences itself also in one's cultural background and occupational setting (Jones, 2012). Thus teaching mathematics well is both appropriate and needed in the modern-day world.

Before examining teaching conceptual understanding of mathematics through hands-on learning, there must be a common understanding of what teaching mathematics entails. As mentioned earlier, mathematics is a prominent part of students' curriculum. Furthermore, oftentimes when an individual mentions mathematics, in an everyday conversation for example, the audience could quite possibly envision a recollection of timed speed drills, a feeling of nausea, a spark of curiosity, etc. which he or she mentally connects with mathematics. R. R. Skemp (1976) explained in "Relational Understanding and Instrumental Understanding" "that *there are two effectively different subjects taught* *under the same name, 'mathematics*'" (p. 22). Skemp had thought that various educators were simply teaching mathematics with varying degrees of quality (1976). Teaching mathematics hinges on the definition of understanding (in the context of mathematics) (Skemp, 1976).

Investigating understanding.

What is mathematical meaning and understanding? Mathematical understanding builds its foundation on mathematical meaning. In the work "The Place of Meaning in the Teaching of Arithmetic," Brownell (1947) specifically explores two expressions: "meanings *of*" and "meanings *for*" (p. 256). The first refers to understanding the workings of the mathematics while the second denotes purpose: using the computations in everyday life (Brownell, 1947). According to Brownell (1947), mathematical meaning comes in various depths. Meaningful mathematics exists, in a child's mind, when there is an application, a goal, to the mathematics (Brownell, 1947).

Closely related to meaning—what something signifies—is understanding—how something works. R. R. Skemp (1976) delineated understanding, differentiating between "'relational understanding'" and "'instrumental understanding'" (based on Stieg Mellin-Olsen's delineations) and providing the implications of each (p. 20). An individual with relational understanding has not only the process but also the reasons behind it solidified while instrumental understanding refers to Skemp's (1976) own past definition: "'rules without reasons'" (p. 20). On the other hand, one educator experienced that rather than individually, strongest are relational and instrumental understanding together (Reason, 2003). Finally, one must note that in mathematics, reasoning is crucial to understanding (Ball & Bass, 2003). In conclusion, in the field of education, the role of understanding in mathematics should not be underestimated.

What is conceptual understanding of mathematics? Therefore, in order to be capable in mathematics, a student needs conceptual understanding, according to the National Research Council (2001). Specifically, the National Research Council argued, "*Conceptual understanding* refers to an integrated and functional grasp of mathematical ideas" (2001, p. 118). Conceptual knowledge and procedural knowledge (which focuses on how to do the arithmetic) is similar to the concepts of relational understanding and instrumental understanding, respectively (Jones, 2012). When a student "understands the meaning and underlying principles of mathematical concepts," he or she has conceptual knowledge in mathematics (Frederick & Kirsch, 2011, p. 94). Knowledge which is conceptual interconnects (Jones, 2012): "Conceptual knowledge requires the learner to be active in thinking about relationships and making connections, along with making adjustments to accommodate the new learning with previous mental structures" (Reys, Suydam, & Lindquist, 1995, p. 21). According to researchers, the order for instilling conceptual and procedural knowledge is first conceptual followed by procedural knowledge (Jones, 2012). Nevertheless, frequently knowing mathematics conceptually is connected strongly with knowing mathematics procedurally (Jones, 2012). In teaching mathematics with meaning, developing a student's conceptual understanding is important.

Why teach understanding? Despite the opposing arguments, there are several reasons why teaching understanding of a concept—specifically in mathematics—is worthwhile. On the one hand, simply learning the algorithm is a less difficult and speedier path to the correct solution and provides visible short-term dividends (Skemp, 1976). In a world focused on getting the correct answer and moving on, teaching meaning can seem unattractive or irrelevant. Although teaching meaning may be more time-consuming, teaching meaning will have a positive effect which builds on itself (Brownell, 1947). Having relational understanding of mathematics eases both applying a principle in an unfamiliar situation and recalling material because students have a broader, more connected outlook (Skemp, 1976). In mathematics but also in other areas, aiming to have understanding of a topic can stand on its own (Skemp, 1976). Relational understanding can even drive a person to discover something new, besides encouraging relational understanding when given unfamiliar content (Skemp, 1976). Also, students can more easily review or relearn mathematical aspects in which they struggle, have less need for back-to-back practice, have problem-solving as the focus of learning, can more easily recognize answers which do not make sense, can use a variety of approaches to finding a solution, and can be more self-sufficient and can feel less daunted by an unfamiliar type of mathematical problem (Brownell, 1947). These are some of the reasons why educators should teach understanding in the area of mathematics.

Furthermore, benefits for specifically teaching conceptual understanding of mathematics abound. The National Council for Teachers of Mathematics (2000), in their work *Principles and Standards for School Mathematics*, argued that in the twenty-first century, students need to have conceptual understanding of mathematics in order to flourish and solve problems as adults in the present changing environment. Also, conceptual understanding of mathematics encourages students to be more independent and confident which evidences itself in students not shrinking back from challenging problems and openness in solving problems differently (NCTM, 2000). When teaching understanding—specifically conceptual understanding—of mathematics, various educational approaches must be considered, so that students may be best served and may learn in the most holistic way possible.

Approaches to the Teaching of Mathematics

The theories investigated provide the context for various approaches to teaching mathematics. The approaches help show how to carry out the goals and views the theories advocate. Specifically, the following approaches will be examined: constructivism and a hands-on approach.

Constructivist. A brief overview of constructivism presents this key underlying philosophy of education evident in many current educational trends. Constructivism is "a theory of learning that asserts that humans construct their own knowledge" (Jones, 2012, p. 34). In the classroom setting, educators would not transmit knowledge but would rather facilitate knowledge (Jones, 2012). Furthermore, constructivism assumes "that children, when faced with problematic arithmetical situations, can develop their own solution methods. The second assumption is that any knowledge that involves carrying out actions or operations cannot be instilled ready-made into children but must, quite literally, be actively built up by them" (J. Piaget's arguments served as the foundation for these statements) (as cited in Steffe & Cobb, 1988, p. vii). Thus constructivism is a pertinent philosophy of education.

Hands-on. A natural outflow of the constructivist philosophy of education, in general and also specifically in teaching mathematics, is hands-on learning. In fact, a hands-on approach plays a crucial role in constructivism (Bhagwanji, 2011). Hands-on learning refers to "learning by doing, or learning in which students are actively engaged in an activity or process," according to B. B. Armbruster (2011, p. 212). Specifically, hands-on instruction involves these aspects noted by Bhagwanji (2011):

> Hands-on curriculum and activities are those in which students touch, move, and experiment with materials in the classroom. As they manipulate objects, children think about the objects' properties and relationships. After several such experiences, children develop "theories" about how things work that can be tested with further manipulation. Children's work with hands-on materials can be assessed and recorded as the children are working, and this data can be analyzed to realize the child's learning progress. (p. 212)

Hands-on learning provides students with interactive learning experiences involving various senses. Furthermore, these hands-on activities engage learners, so that they do not just passively listen to a lecture, for example, but instead personally have to interact with the material. Frequently hands-on learning requires students to solve problems. One way to engage students in hands-on learning is to use manipulatives. Using manipulatives can greatly aid students in developing their conceptual understanding. In responding to the need for teaching conceptual understanding of mathematics, hands-on learning provides an appropriate pedagogical approach.

Research

Research proves a theory's viability and how well one can implement that specific theory. Before evaluating the various research, one must note that research related to evaluating the teaching of understanding in mathematics can present difficulties in eliminating "the impact of any particular variable such as curricular goals, teacher knowledge, available classroom materials, and contextual surround" (Schoenfeld, 2008,

p. 125). Nevertheless, examining research related to conceptual understanding via a hands-on approach is worthwhile. This brief presentation focuses specifically on implementation, tools, and evaluation. Using a hands-on approach provides an effective way to implement the teaching of conceptual understanding.

Studies

Implementation. Hands-on learning to teach conceptual understanding can be implemented in a classroom. Gürbüz, Çatlioğl, Bìrgìn, and Erdem, (2010) researched the effect of applying a hands-on learning approach on students' learning: "Activities, designed to present abstract mathematical expressions in a concrete and visual way help students develop creative thinking and imagination," according to P. W. Thompson (1992) (as cited in Gürbüz et al., 2010, p.1054). In this study, the mathematical concept of probability served as the backdrop for the researchers' study (Gürbüz et al., 2010).

The researchers' study (Gürbüz et al., 2010) involving fifty fifth grade students' probability learning and understanding divided students into two equal groups: half of the fifty participated in an experimental group and the rest in a control group (Gürbüz et al., 2010). The researchers used a fifteen-question Conceptual Development Test to gather data (Gürbüz et al., 2010). Ideas related to "sample space (SS), probability of an event (PE), [and] probability comparisons (PC)" entail the three areas the test covered (Gürbüz et al., 2010, p. 1057). As a pre-assessment and point of comparison, first all fifty of the students took the pre-test tailored to the previously mentioned topics (Gürbüz et al., 2010). In evaluating the results of the pre-test, the researchers revealed that the two groups of students showed virtually equal understanding related to probability: SS, PE, or PC (Gürbüz et al., 2010). Next in the study came the teaching phase (Gürbüz et al., 2010).

Students in the experimental group, in small groups of three to four students, participated in a hands-on activity in which students not only experimented but also communicated with each other (Gürbüz et al., 2010). The researchers argued that students in the experimental group "have constructed their knowledge more meaningfully and showed a better cognitive development" (Gürbüz et al., 2010, p. 1058). In the hands-on setting, the teacher has a different role from the traditional role where he or she lectures and explains (Gürbüz et al., 2010). Also, students in the hands-on setting "have become more active, improved their knowledge, questioned the knowledge they got, and . . . were able to explain what they know instead of being passive during class" (Gürbüz et al., 2010, p. 1058).

On the other hand, the teacher was the focus during instruction—presented in a written, that is, visual, and auditory manner—in the control group (Gürbüz et al., 2010). Although the teacher provided students the opportunity to ask questions and answered questions, students were passive in this learning process (Gürbüz et al., 2010): sitting, they took notes (Gürbüz et al., 2010). In this lesson, the teacher lectured for about three-fourths of the total lesson time (Gürbüz et al., 2010). Then students answered questions (Gürbüz et al., 2010).

Students in both groups took a post-test a month after the two teaching approaches had been implemented (Gürbüz et al., 2010). Researchers found that out of the two groups, the experimental group had better conceptual understanding (Gürbüz et al., 2010). The experimental group's students' understanding of the concepts addressed was greatly influenced by the instruction given in this group: by the hands-on approach

(Gürbüz et al., 2010). Overall, this study shows that because students could interact with the material on a personal level—such as being involved in experimentation and communication—the hands-on learning approach effected learning related to probability (Gürbüz et al., 2010). So, students physically experimenting is another benefit of hands-on learning (Gürbüz et al., 2010). This study shows the effects of using a hands-on learning approach.

Tools. In employing a hands-on approach to teach conceptual understanding, one useful tool is using manipulatives. Manipulatives, or "concrete objects, . . . are widely used in mathematics because they help students learn mathematical concepts and skills" (Tipps, Johnson, & Kennedy, 2011, p. 59). Manipulatives aid students in noting and communicating with others their understanding of mathematical thoughts and also in making mental processes and thoughts more refined (Özgün-Koca & Edwards, 2011):

> Kaput (1995) differentiates the relationships between mental operations and physical observations . . . . When one moves from mental operations to physical operations, 'one has cognitive content that one seeks to externalize for purposes of communication or testing for viability' (p. 140). On the other hand, in moving from physical observations to mental operations, 'processes are based on an intent to use some existing physical material to assist one's thinking' (p. 140) . . . . (as cited in Özgün-Koca & Edwards, 2011, p. 392)

Thus the use of manipulatives in promoting conceptual understanding supports a hands-on approach.

Özgün-Koca and Edwards (2011) implemented a research study in which algebra students in eighth grade used a physical manipulative and a virtual manipulative in learning about lines of best fit. The first manipulative used in this study was physical spaghetti, which students used to determine the line of best fit for points that were plotted for them (Özgün-Koca & Edwards, 2011). Then students used virtual spaghetti to mark the line of best fit on a TI-Nspire (graphing calculator), which not only used line segments to show the distances between the virtual spaghetti and the already plotted points but provided an error calculation as well (Özgün-Koca & Edwards, 2011).

In this study, in order to evaluate the use of the physical and virtual manipulatives, questions were asked, with answers given by a total of forty-one students (Özgün-Koca & Edwards, 2011). The electronic form of the teaching tool presented itself as the favored choice for most—approximately 85%—of the students who answered the research questions (Özgün-Koca & Edwards, 2011). The movability as well as interactivity of the digital manipulative's spaghetti both ranked high as the favored aspects (Özgün-Koca & Edwards, 2011): "Some students mentioned that manipulating the virtual spaghetti helped their understanding[,] and other students found it more fun" (p. 398). On the other hand, Özgün-Koca & Edwards (2011) found that the physical version had won the favor of only about 15% of the students who responded to the questions. These students "found it easier and . . . liked the hands-on experience" (Özgün-Koca & Edwards, 2011, p. 398). Also related to using a hands-on approach, "even though the majority preferred the virtual spaghetti, they stated that the part they liked about the real spaghetti was the hands-on experience and being able to touch the spaghetti strand" (Özgün-Koca & Edwards, 2011, p. 398). The researchers argued that because more than one way in which students learn was addressed in using the physical and digital representations together, students actually benefited more than if they had used just one or the other approach (Özgün-Koca & Edwards, 2011). Furthermore, the researchers of this study argued that further research is required (Özgün-Koca & Edwards, 2011): For example, which manipulative should teachers present first (Özgün-Koca & Edwards, 2011)? What proportion of student interaction should be with the physical manipulative compared to the digital (Özgün-Koca & Edwards, 2011)? Or how ought the non-physical manipulative be laid out, and furthermore, what extent of similarity should exist between the two types (Özgün-Koca & Edwards, 2011)? Thus, the value of using manipulatives, key elements and hallmarks of hands-on learning, cannot be understated.

Evaluation. If teachers cannot identify when students have conceptual understanding of a mathematical area, and when not, then even the best learning activities prove quite ineffective. Identifying when a student does or does not have conceptual understanding can be quite difficult, especially compared to grading a traditional test where one solely checks for a correct numerical solution. Therefore examining the following investigation will help provide insight into the evaluation aspect of implementing teaching conceptual understanding with a hands-on approach.

Researchers conducted a study involving fifty-four prospective educators studying elementary education (at the undergraduate level) and focused on not only how identifying conceptual understanding in others reflects prospective educators' own conceptual understanding but also how teaching identifying conceptual understanding affects this skill in pre-service educators (Bartell, Webel, Bowen, & Dyson, 2013):

> Analysis of PSTs' [prospective teachers'] content knowledge [researchers' clarifying term for conceptual understanding of the PSTs] shows that all . . .

> demonstrated *some evidence* of conceptual understanding of subtraction of decimals, most demonstrated *some evidence* of conceptual understanding of comparison of fractions, but few demonstrated evidence of conceptual understanding of multiplication of fractions. (p. 68)

In examining how well pre-service educators evaluated students' work, both students indeed having conceptual understanding and students actually only having procedural knowledge, at the beginning of this study, were classified as having conceptual understanding by many prospective educators (whether the prospective educator had a strong grasp of that mathematical concept or not) (Bartell et al., 2013). Furthermore, in checking for conceptual understanding in students (by examining their work), merely in the area of fraction comparison did pre-service teachers' own conceptual understanding actually aid them in this evaluation (interestingly, a misconception on the student side solely presented itself in comparing fractions) (Bartell et al., 2013).

Next, since the future teachers were a part of a mathematics class at a university, the intervention for these prospective teachers took place in the context of their course, specifically taking place in three times they met (Bartell et al., 2013). Three course sections (with one additional section as a control group) participated in the intervention (Bartell et al., 2013). In the first part of the intervention, guiding prospective teachers to what points to a student's conceptual understanding, pre-service educators had been assigned a specified video to watch on a mathematics lesson for first grade, followed by prospective teachers then interacting with the presented material in class (Bartell et al., 2013). The prospective teachers interacted in both a group and entire classroom setting, "with instructors highlighting that good teacher explanations do not mean that students understand, and sometimes activities meant to give conceptual understanding can become proceduralized in practice" to the entire class (Bartell et al., 2013, p. 60). The focus on indications of students' mathematical understanding continued in the scrutiny which pre-service teachers' gave the video lesson once more (Bartell et al., 2013).

The intervention of Bartell et al. (2013) continued: for adding the numbers 63.7 and 49.8, nine answers which would commonly be given stood under the future teachers' analysis (Bartell et al., 2013). The future teachers "discussed what each child knew about the addition of decimal quantities" (Bartell et al., 2013, p. 61). In this second part of the intervention, the prospective educators also did two categorization activities and dialogued as a class, one time in which the course instructor focused not only on classification of understanding but also what would indicate that a student truly understands conceptually (for example, whether certain depictions, or even certain vocabulary terms, are adequate) (Bartell et al., 2013).

In respect to the effectiveness of the intervention conducted, this study showed both effectiveness and ineffectiveness (Bartell et al., 2013). For example, prospective teachers tended to depart "from evaluating responses with conceptual features and procedural solutions as evidence of conceptual understanding for the subtraction of decimals content" as a result of the intervention (Bartell et al., 2013, p. 71):

> While notable numbers of PSTs also moved away from evaluating such responses as evidence of conceptual understanding for the multiplication of fractions content, many PSTs still saw these responses as evidence of conceptual understanding after the intervention. This was also true for the analyses of procedural responses in the comparison of fractions content. Further, in the areas

> where there was growth in PSTs' analyses of children's mathematical work, we also saw that PSTs tended to become critical of children's responses, suggesting that responses with conceptual features or procedural solutions were evidence that the child did *not* understand the mathematics. (Bartell et al., 2013, p. 71)

Therefore, educators need more than merely understanding of the conceptual aspects of a mathematics topic in order to effectively evaluate a student's response as showing conceptual understanding or not (Bartell et al., 2013). In conclusion, prospective teachers should develop their own conceptual understanding (Bartell et al., 2013). Focusing specifically on how procedural knowledge and conceptual understanding are different, practice is also important in the endeavor to determine whether a student has either knowledge at the procedural or conceptual level (Bartell et al., 2013). Educators need to be skilled in evaluating a student's conceptual understanding, besides teaching it via a hands-on activity.

Hands-On Activities

Besides presenting research on teaching conceptual understanding through hands-on activities—specifically focusing on implementation, tools, and evaluation, presenting a practical hands-on activity for teaching conceptual understanding provides added value to the overall investigation. The research presented in studies can oftentimes seem disconnected from an educator's own experiences and classroom. Thus, the following hands-on activity aims to provide a practical example, bridging the gap between educational research and actual classroom implementation.

Geometry. An area in mathematics which lends itself easily to hands-on activities is geometry. Students begin to develop their conceptual understanding of geometry concepts as a young child and develop their understanding throughout their schooling. Specifically, in upper elementary grades, students lay the foundation for geometry in middle and high school. Engaging students in a hands-on activity in geometry allows students to grow their conceptual understanding of the topic.

In the subsequent rephrased activity, designed for fourth through sixth graders, students individually "find a pattern associated with the number of sides and the number of diagonals in polygons" (Tipps et al., 2011, p. 452). First, using a geoboard, students form a "triangle, quadrilateral, [and] pentagon" (Tipps et al., 2011, p. 452). Then students can indicate the polygons' diagonals with rubber bands (Tipps et al., 2011). A table with these categories—the polygon's name (all provided), number of sides/angles (only the first four provided), and number of diagonals (the first two provided)—provides students the place for them to record their findings (Tipps et al., 2011). Using the geoboards and rubber bands, polygons with six sides through about eight sides (in sequential order) should be created, with students recording their findings (how many angles and diagonals each polygon has) (Tipps et al., 2011). A pattern should be detectable by around the time students create an octagon and should, via making educated guesses, be used to complete the part of the table for noting diagonals (Tipps et al., 2011). Then, "ask students to explain how they found the number of diagonals in the figure with ten sides" and to "develop a number sentence (formula) for finding the number of diagonals in a figure with any number of sides" (Tipps et al., 2011, p. 452). The geometry activity allows students to discover the algorithm which relates polygons' sides and diagonals (Tipps et al., 2011) in a sensory, hands-on way. The geometry activity presented provides a realistic example of a hands-on activity which could help develop students' conceptual understanding of mathematics.

Discussion

Interpretation

Studies. The various studies presented provide different insights, specifically related to implementation, tools, and evaluation, into the teaching of conceptual understanding via a hands-on approach, while the hands-on activity provided an example of teaching conceptual understanding which educators can actually implement in the classroom. Just presenting the research is not enough—one must interpret the research, finding out what it means. Thus, the various studies and the hands-on activity will be discussed.

The first study presented—investigating the use of a hands-on approach in students' learning of probability—gives an inside-look into teaching conceptual understanding of mathematics in an actual classroom setting (Gürbüz et al., 2010). The sample size for Gürbüz et al.'s 2010 study was not very large. Further investigation into practical techniques effective in implementing a hands-on approach in teaching probability could be a beneficial study to educators.

Secondly, the study on the use of physical versus virtual manipulatives (Özgün-Koca & Edwards, 2011) demonstrates the value of using manipulatives. As mentioned earlier, manipulatives often are central in using a hands-on approach, despite the fact that "some people believe that manipulatives are just for younger children or for slow learners" (Tipps et al., 2011, p. 88). Özgün-Koca and Edwards' 2011 study provides a viable example demonstrating that this negative view of manipulatives, which Tipps et al. (2011) mentioned, is inaccurate. Furthermore, even virtual manipulatives could still be included in hands-on activities, for, as mentioned earlier, hands-on learning refers to "learning by doing, or learning in which students are actively engaged in an activity or process" (Armbruster, 2011, p. 212). Furthermore, as stated earlier, "hands-on curriculum and activities are those in which students touch, move, and experiment with materials in the classroom. As they manipulate objects, children think about the objects' properties and relationships" (Bhagwanji, 2011, p. 212). Virtual manipulatives not only can be used in hands-on activities but fit in well with the trend and push for incorporating the use of technology in the classroom. Although this trend and push is ever increasing, some schools may not have the necessary technological instruments to include virtual manipulatives in the mathematics classroom. Furthermore, whether there are many or not so many quality virtual manipulatives available for educators' use must be determined—as well as the accessibility to these virtual manipulatives. Overall, the discussion of physical and virtual manipulatives corresponds to the current educational trend of increasing the use of technology in the classroom. Incorporating manipulatives supports a hands-on approach to teaching conceptual understanding of mathematics.

The last study, on evaluating a students' level of conceptual understanding, provides a much-needed insight into evaluating conceptual understanding (Bartell et al., 2013). Amidst the call for educators to help develop students' conceptual understanding in mathematics, one should not neglect the other side necessary in implementing this goal: Educators need to correctly evaluate a student's conceptual understanding. Clearly, this provides some challenges (Skemp, 1976). Firstly, the educator cannot know exactly what the student is thinking, so the student must communicate his or her thinking in a verbal (Skemp 1976), written, or other manner. Herein lies the first challenge: One must make sure that the student is actually communicating what he or she knows. Then, according to Bartell et al. (2013), to evaluate whether or not the student has proper conceptual understanding, the educator needs to have conceptual knowledge and practice. In addition, Bartell et al.'s 2013 study used pre-service teachers in their investigations. Amidst the push for teaching conceptual understanding, in order to accurately evaluate students' levels of conceptual understanding, the necessary training (Bartell et al., 2013) and physical tools of reference are necessary for current educators. The third study (Bartell et al., 2013) evaluated provides great insight into the issue of evaluating a student's conceptual understanding and can be a starting point for further research—for example, on the educators' training in evaluating conceptual understanding (Bartell et al., 2013). In conclusion, the third study (Bartell et al., 2013) demonstrates the need for teachers to be able to effectively evaluate students' learning.

Hands-on activities. The geometry mathematics activity provides a workable and realistic example of a hands-on activity. The activity's creators actually noted that its use is in assessing students (Tipps et al., 2011), but this activity could be modified for other situations such as guided practice, independent practice, or centers. Instead of just presenting the algorithm for finding the number of diagonals in an *n*-sided polygon, students can interact with the material in a physical way, for example by touching, and seeing, and creating the diagonals (Tipps et al., 2011). Fostering students' conceptual understanding in mathematics, hands-on activities allow students to interact with mathematics concepts at a deeper level, not just providing superficial knowledge of how to find the correct answer but conceptual understanding—of diagonals in a polygon, for example, (Tipps et al., 2011). Thus hands-on activities provide educators opportunities to grow students' conceptual understanding.

Implications

The research and goals towards using a hands-on approach in teaching conceptual understanding in mathematics cannot remain in the realm of academia but must be applied to the average classroom. In addition, in order to help prove the effectiveness of hands-on learning on conceptual understanding, more research needs to be conducted with this specific intersection of educational approach and goal. However, even the research already available supporting the use of a hands-on approach in teaching conceptual understanding of mathematics has several implications for both educators and students.

Although teachers and prospective teachers may know the value of teaching mathematics in a way which promotes understanding, some potential hindrances exist as well, which Skemp detailed in his 1976 work. Specifically, teaching mathematics using a hands-on approach may seem unrealistic. With the current pressure related to standardized test scores (Skemp explored tests but in a slightly different light (1976)), diversity in areas such as learning and culture, large class sizes, and more, teaching conceptual understanding of mathematics using a hands-on approach may seem like a great idea but may end there—as an idea. This does not have to be the case. Start small. The activities do not need to be extravagant. Educators can discover a plethora of hands-on activities through resources on the internet. Overall, the benefits of using hands-on activities in teaching conceptual understanding show the value of teaching conceptual understanding via a hands-on approach.

A hands-on approach in teaching conceptual understanding of mathematics implication also affects both teachers and students in that using a hands-on approach supports differentiated instruction. Using strategies that are hands-on can provide opportunities for various grouping options, which ties together well with differentiation. On the one hand, students learn “as individuals,” that is, at different paces, and so educators will need to tailor activities to correspond to students’ instructional levels (Marks, Purdy, Kinney, & Hiatt, 1975, p. 323). But also, students learn in the setting of community (Askew, 2012). Given the various interconnections between hands-on activities and differentiation, hands-on activities provide a viable option for encouraging the development of conceptual understanding in a classroom that embraces differentiated instruction.

Conclusion

In conclusion, using a hands-on approach provides an effective way to teach conceptual understanding of mathematics. Conceptual understanding of mathematics lies at the core of mathematics, and using a hands-on approach to develop students’ conceptual understanding naturally flows out of current educational trends and research. Furthermore, conceptual understanding and hands-on activities encourage a more holistic education. The value of teaching conceptual understanding of mathematics by means of a hands-on approach is tremendous.

No matter how powerfully nor profusely research proves the value of a certain aspect of mathematics that should be taught or the value of a certain approach, the research in and of itself will not effect anything. Teachers, parents, students themselves, schools, community leaders, school administrators, educational organizations, and others need to communicate that they also see the value: specifically that they value using a hands-on approach in teaching conceptual understanding of mathematics. A theory can be great and can prove to even hold up in research, but unless you and I act on this knowledge, the scholarly efforts are laid ineffective. What will aid in increasing the use of a hands-on approach in teaching conceptual understanding of mathematics in the classroom is each person acting within his or her realm of influence.

References

Armbruster, B. B. (2011). Hands-on learning. In J. W. Collins III and N. P. O'Brien (Eds.), *The Greenwood dictionary of education* (2nd ed.) (p. 212). Santa Barbara, CA: Greenwood.

Askew, M. (2012). *Transforming primary mathematics*. New York, NY: Routledge.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to* Principles and Standards for School Mathematics (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Bartell, T. G., Webel, C., Bowen, B., & Dyson, N. (2013). Prospective teacher learning: Recognizing evidence of conceptual understanding. *Journal of Mathematics Teacher Education, 16*(1), 57-79. doi:10.1007/s10857-012-9205-4

Bhagwanji, Y. (2011). Hands-on curriculum/activities. In J. W. Collins III and N. P. O'Brien (Eds.), *The Greenwood dictionary of education* (2nd ed.) (p. 212). Santa Barbara, CA: Greenwood.

Brownell, W. A. (1947). The place of meaning in the teaching of arithmetic. *The Elementary School Journal, 47*(5), 256-265. Retrieved from http://www.press.uchicago.edu/ucp /journals/journal/esj.html

Devlin, K. J. (1994). *Mathematics: The science of patterns: The search for order in life, mind, and the universe*. New York, NY: Scientific American Library.

Frederick, M. F., & Kirsch, R. L. Conceptual mathematics knowledge. In J. W. Collins III and N. P. O'Brien (Eds.), *The Greenwood dictionary of education* (2nd ed.) (p. 94). Santa Barbara, CA: Greenwood.

Gürbüz, R., Çatlioğlu, H., Bìrgìn, O., & Erdem, E. (2010). An investigation of fifth grade students' conceptual development of probability through activity based instruction: A quasi-experimental study. *Educational Sciences: Theory & Practice, 10*(2), 1053-1068. Retrieved from http://edam.com.tr/

Jones, J. C. (2012). *Visualizing elementary and middle school mathematics methods*. Hoboken, NJ: Wiley.

Marks, J. L., Purdy, C. R., Kinney, L. B., & Hiatt, A. A. (1975). *Teaching elementary school mathematics for understanding* (4th ed.). New York, NY: McGraw-Hill.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. J. Carpenter & S. Gorg (Eds.). Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Washington, DC: National Academy Press.

Özgün-Koca, S. A. & Edwards, T. (2011). Hands-on, minds-on or both? A discussion of the development of a mathematics activity by using virtual and physical manipulatives. *Journal of Computers in Mathematics and Science Teaching, 30*(4), 389-402. Retrieved from http://www.aace.org/pubs/jcmst/

Piaget, J. (1970). *Science of education and the psychology of the child*. (D. Coltman, Trans.). New York, NY: Orion Press.

Reason, M. (2003). Relational, instrumental and creative understanding. *Mathematics Teaching,* (184), 5-7. Retrieved from http://www.atm.org.uk /Mathematics-Teaching-Journal-Archive

Reys, R. E., Suydam, M. N., & Lindquist, M. M. (1995). *Helping children learn mathematics* (4th ed.). Needham Heights, MA: Allyn and Bacon.

Romberg, T. A., & Kaput, J. J. (1999). Mathematics worth teaching, mathematics worth understanding. In E. Fennema & T. A. Romberg. *Mathematics classrooms that promote understanding* (pp. 3-17). Mahwah, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (2008). Mathematics for understanding. In L. Darling-Hammond, B. Barron, P. D. Pearson, A. H. Schoenfeld, E. K. Stage, T. D. Zimmerman, G. N. Cervetti, & J. L. Tilson (Authors), *Powerful learning: What we know about teaching for understanding* (pp. 113-150). San Francisco, CA: John Wiley & Sons.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26. Retrieved from http://www.atm.org.uk /Mathematics-Teaching-Journal-Archive

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies* (E. von Glasersfeld, Contributor) (pp. vii-xi). New York, NY: Springer-Verlag.

Tipps, S., Johnson, A., & Kennedy, L. M. (2011). *Guiding children's learning of mathematics* (12th ed.). Belmont, CA: Wadsworth.